THE GEOMETRY OF EIGHT POINTS IN PROJECTIVE SPACE: REPRESENTATION THEORY, LIE THEORY, DUALITIES

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Abstract. This paper deals with the geometry of the space (GIT quotient) M_8 of 8 points in $\mathbb{P}^1$, and the Gale-quotient N_8' of the GIT quotient of 8 points in $\mathbb{P}^3$.

The space M_8 comes with a natural embedding in $\mathbb{P}^{13}$, or more precisely, the projectivization of the $\mathfrak{S}_8$-representation $V_{4,4}$. There is a single $\mathfrak{S}_8$-skew cubic $\mathcal{C}$ in $\mathbb{P}^{13}$. The fact that M_8 lies on the skew cubic $\mathcal{C}$ is a consequence of Thomae's formula for hyperelliptic curves, but more is true: M_8 is the singular locus of $\mathcal{C}$. These constructions yield the free resolution of M_8, and are used in the determination of the "single" equation cutting out the GIT quotient of n points in $\mathbb{P}^1$ in general [HMSV4].

The space N_8' comes with a natural embedding in $\mathbb{P}^{13}$, or more precisely, $\mathbb{P}V_{2,2,2,2}$. There is a single skew quintic $\mathcal{Q}$ containing N_8', and N_8' is the singular locus of the skew quintic $\mathcal{Q}$.

The skew cubic $\mathcal{C}$ and skew quintic $\mathcal{Q}$ are projectively dual. (In particular, they are surprisingly singular, in the sense of having a dual of remarkably low degree.) The divisor on the skew cubic blown down by the dual map is the secant variety $\mathrm{Sec}(M_8)$, and the contraction $\mathrm{Sec}(M_8) \dashrightarrow N_8'$ factors through N_8 via the space of 8 points on a quadric surface. We conjecture (Conjecture 1.1) that the divisor on the skew quintic blown down by the dual map is the quadrisecant variety of N_8' (the closure of the union of quadrisecant *lines*), and that the quintic $\mathcal{Q}$ is the trisecant variety. The resulting picture extends the classical duality in the 6-point case between the Segre cubic threefold and the Igusa quartic threefold.

We note that there are a number of geometrically natural varieties that are (related to) the singular loci of remarkably singular cubic hypersurfaces, e.g. [CH], [B], etc.

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1. Introduction

This note discusses the geometry of the spaces

$$M_8 := (\mathbb{P}^1)^8/\!/\mathrm{SL}(2), \quad N_8 := (\mathbb{P}^3)^8/\!/\mathrm{SL}(4) \quad \text{and} \quad (\mathbb{P}^5)^8/\!/\mathrm{SL}(6),$$

each the GIT quotient of 8 points in projective space with respect to the "usual" linearization $\mathcal{O}(1,\dots,1)$. For each of these quotients Q, let $R_\bullet(Q)$ be the corresponding (graded) ring of invariants. (Coble–)Gale duality gives a canonical isomorphism between the first and third, via a canonical isomorphism of the graded rings of invariants. Gale duality gives an involution on N_8, through an involution of its underling graded ring $R_\bullet(N_8)$. (An explicit description of Gale duality in terms of tableaux due to [HM] is given in the proof of Proposition 8.2.) Our goal is to study and relate M_8 and N_8 (and the Gale-quotient N_8' of N_8) and their extrinsic geometry. The key constructions are dual hypersurfaces $\mathcal{C}$ and $\mathcal{Q}$ in $\mathbb{P}^{13}$ of degrees three and five respectively; for example, $M_8 = \mathrm{Sing}(\mathcal{C})$ and $N_8' = \mathrm{Sing}(\mathcal{Q})$ (§3.4). The partial derivatives of $\mathcal{C}$ (we sloppily identify hypersurfaces and their underlying equations), which cut out M_8, will be referred to as "the 14 quadratic relations"; they span an irreducible $\mathfrak{S}_8$-representation (of type $2+2+2+2$, see Proposition 2.2), and up to symmetry there is only one quadratic relation (given in appropriate coordinates by a simple binomial relation (2)).

In [HMSV4], we give *all* relations among generators of the graded rings for $(\mathbb{P}^1)^n/\!/\mathrm{SL}(2)$, with *any* linearization. In each case the graded rings are generated in one degree, so the quotients come with a natural projective embedding. The general case reduces to the linearizations 1^n, with n even. In this 1^n case, with the single exception of $n = 6$, there is (up to $\mathfrak{S}_n$-symmetry) a single quadratic equation, which is binomial in the Kempe generators (Specht polynomials). The quadratic for the case $n \geq 8$ is pulled back from the (unique up to symmetry) $n = 8$ quadratic discussed below, which forms the base case of an induction. We indicate in §1.5 how the only case of smaller n with interesting geometry ($n = 6$, related to Gale duality, and projective duality of the Segre cubic and the Igusa quartic) is also visible in the boundary of the structure we describe here. Thus various beautiful structures of GIT quotients of n points on $\mathbb{P}^1$ are all consequences of the geometry in the 8-point space M_8 discussed in this paper.

The main results are outlined in §1.2, and come in three logically independent parts. The first deals with the relationship between M_8 and N_8. The second deals solely with M_8, and the third with N_8.

(A) In §3, we describe the intricate relationship between M_8 and N_8, summarized in Figure 1. We note that this section does not use the fact that the ideal cutting out M_8 is *generated* by the 14 quadratic relations (established by pure thought in (B)), only that it lies in the *intersection* of the 14 quadratic relations (§3.1). We also do not use that N_8' is the Gale-quotient of N_8 (established in (C)), only that N_8' corresponds to the subring of the

ring of invariants of $(\mathbb{P}^3)^8/\!/\mathrm{SL}(4)$ generated in degree 1. (For this reason we take this as our initial definition of N_8'.)

(B) In §4–6, we give a Lie-theoretic proof of the fact that there are no linear syzygies among the 14 quadratic relations (Theorem 4.1). This (in combination with results of [HMSV4]) gives a pure thought proof that the 14 quadratic relations generate the ideal of M_8, the base case of the main induction of [HMSV4] (Corollary 4.2). This was known earlier via computer calculation by a number of authors (Maclagan, private communication; Koike [Koi]; and Freitag and Salvati-Manni [FS2]), but we wished to show the structural reasons for this result in order to make the main theorem of [HMSV4] (giving all relations for all GIT quotients of $(\mathbb{P}^1)^n/\!/\mathrm{SL}(2)$) computer-independent. (Strictly speaking, in [HMSV4], computers were used to deal with the character theory of small-dimensional $\mathfrak{S}_6$-representations, but this could certainly be done by hand with some effort.) In §7, we use the absence of linear syzygies to determine the graded free resolution of (the ring of invariants of) M_8.

(C) In the short concluding section §8, we verify with the aid of a computer that the subring $R_\bullet(N_8')$ of $R_\bullet(N_8)$ generated in degree 1 is indeed the ring of Gale invariants, and that the skew quintic is the *only* skew quintic relation in both N_8' and N_8. This is done by verifying that $R_\bullet(N_8)$ is generated in degrees one and two, and determining the actions of $\mathfrak{S}_8$ and Gale duality on these generating sets.

To be clear on the use of computer calculation (as opposed to pure thought): in §3, we use a computer only to intersect two curves in $\mathbb{P}^2$; in §4–7, computers are not used; and computer calculation is central to §8.

We describe other manifestations of the ring of invariants of M_8 in §1.3. Miscellaneous algebraic results about M_8 that may be useful to others are given in §1.4. We sketch how the beautiful classical geometry of the six point case is visible at the boundary in §1.5. The justifications of the statements made in §1.2 are given in the rest of the paper.

1.1. **Notation.** *In general, we work over a field* $\mathbf{k}$ *of characteristic* 0. Most statements work away from a known finite list of primes, so we occasionally give characteristic-specific statements. For a partition λ of n, we write V_λ for the corresponding irreducible representation of $\mathfrak{S}_n$. The $\mathfrak{S}_8$-representations important for us are the trivial (V_8) and sign ($\mathrm{sgn} := V_{1^8}$) representations, and the two 14-dimensional representations $V_{4,4}$ and $V_{2,2,2,2}$. The latter two are skew-dual: $V_{4,4} \otimes \mathrm{sgn} \cong V_{2,2,2,2}$. The representation $V_{3,1,1,1,1,1}$ appears in §8.

1.2. **Main constructions (see Figure 1).** All statements made here will be justified later in the paper.

The ring $R_\bullet(M_8) = \bigoplus_k \Gamma(\mathcal{O}_{\mathbb{P}^1}(k)^{\boxtimes 8})^{\mathrm{SL}(2)}$ is generated in degree 1 (Kempe's 1894 theorem, see for example [HMSV1, Thm. 2.3]), and $\dim R_1(M_8) = 14$ (§2.1). We thus have a natural closed immersion $M_8 \hookrightarrow \mathbb{P}^{13}$. By Schur–Weyl duality or a comparison of tableaux descriptions (§2.1–2.2), $R_1(M_8)$ carries the irreducible $\mathfrak{S}_8$-representation $V_{4,4}$.

The ideal of relations of M_8,

$$I_\bullet(M_8) := \ker(\mathrm{Sym}^\bullet R_1(M_8) \to R_\bullet(M_8)),$$

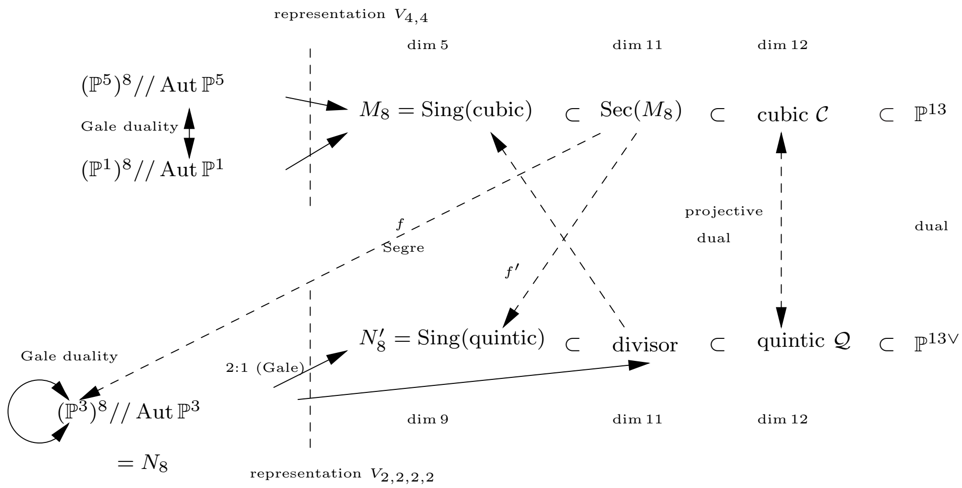

FIGURE 1. Diagram of interrelationships.

is generated by 14 quadratic relations (Corollary 4.2, known earlier by computer calculation as described in (B) above). There is (up to multiplication by non-zero scalar) a unique skew-invariant cubic (an element of $\operatorname{Sym}^3 R_1(M_8)$, Proposition 2.2(a)). We call this cubic the *skew cubic* $\mathcal{C}$, and by abuse of notation we call the corresponding hypersurface $\mathcal{C}$ as well. The fact that M_8 lies on the skew cubic $\mathcal{C}$ is a consequence of Thomae's formula for hyperelliptic curves. (We thank Sam Grushevsky explaining this to us.) But more is true — the fivefold M_8 is the singular locus of $\mathcal{C}$ in a strong sense: $I_\bullet(M_8)$ is the Jacobian ideal of $\mathcal{C}$ — the 14 partial derivatives of the skew cubic $\mathcal{C}$ generate $I_\bullet(M_8)$ and are of course the 14 quadratic relations described above (§3.1). (In fact, this result holds away from characteristic 3. In characteristic 3, the Euler formula yields a linear syzygy among the 14 quadratic relations, and the skew cubic $\mathcal{C}$ can be taken as the remaining generator of the ideal.)

The ring $R_\bullet(N_8) = \bigoplus_k \Gamma(\mathcal{O}_{\mathbb{P}^3}(k)^{\boxtimes 8})^{\mathrm{SL}(4)}$ is generated in degree 1 and 2 (Proposition 8.1), and $\dim R_1(N_8) = 14$. As an $\mathfrak{S}_8$-module, $R_1(N_8)$ is irreducible of type $V_{2,2,2,2}$ (as with $R_1(M_8)$, by Schur–Weyl duality, §2.2, or by direct comparison of the tableaux description). The Gale-invariant subalgebra is the subalgebra $R_\bullet(N'_8) \subset R_\bullet(N_8)$ generated by $R_1(N_8)$. More precisely: we *define* the graded ring $R_\bullet(N'_8)$ as the subalgebra of $R_\bullet(N_8)$ generated in degree 1 (i.e., by $R_1(N_8)$), and *define* $N'_8 = \operatorname{Proj} R_\bullet(N'_8)$, then *show* (in Proposition 8.2) that $R_\bullet(N'_8)$ is the Gale-invariant subalgebra.

Bezout's theorem implies $\operatorname{Sec}(M_8) \subset \mathcal{C}$: restricting the cubic form $\mathcal{C}$ to any line joining two distinct points of M_8 yields a cubic vanishing to order 2 at those two points (as $M_8 = \operatorname{Sing}\mathcal{C}$), so this cubic must vanish on the line. The secant variety $\operatorname{Sec}(M_8)$ has dimension 11 as one would expect (Corollary 3.9(a)), and is thus a divisor on the 12-fold $\mathcal{C}$.

Let

$$I_\bullet(N'_8) := \ker\left(\operatorname{Sym}^\bullet(R_1(N'_8)) \to R_\bullet(N'_8)\right)$$

be the ideal of relations of N_8'. By comparing the readily computable $\mathfrak{S}_8$-representations $\mathrm{Sym}^5(R_1(N_8'))$ and $R_5(N_8')$, we find that there is a skew quintic *relation* $\mathcal{Q}$ in $I_5(N_8')$ (Proposition 2.2(c) and Theorem 3.15; uniqueness is shown later in Proposition 8.3). Furthermore, the ninefold N_8' is the singular locus of $\mathcal{Q}$ (Theorem 3.18).

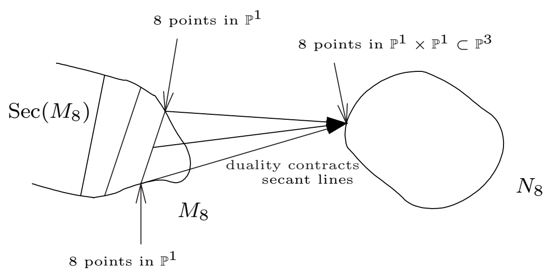

FIGURE 2. The contraction of the secant variety of $M_8 = (\mathbb{P}^1)^8//\mathrm{SL}(2)$ to $N_8 = (\mathbb{P}^3)^8//\mathrm{SL}(4)$.

Moreover, $\mathcal{C}$ and $\mathcal{Q}$ are dual hypersurfaces in the sense of projective geometry (Theorem 3.15).

$$\begin{tikzcd} \mathcal{C} \arrow[r, dashed, bend right, "D"'] & \mathcal{Q} \arrow[l, dashed, bend right, "D'"'] \end{tikzcd}$$

Every secant line $\ell = \overline{pq}$ to M_8 (where $p, q \in M_8$, $p \neq q$) is contracted by the dual map $D : \mathcal{C} \dashrightarrow \mathcal{Q}$: the dual map is given by the 14 partial derivatives of $\mathcal{C}$; their restrictions to ℓ are 14 quadratic relations vanishing at the same two points p, q, so they are the same up to scalar. Thus $\mathrm{Sec}(M_8)$ is contained in the exceptional divisor of the dual map $D : \mathcal{C} \dashrightarrow \mathcal{Q}$, and in fact is the entire exceptional divisor (§3.4). Thus the dual map D contracts $\mathrm{Sec}(M_8)$ to $\mathrm{Sing}(\mathcal{Q}) = N_8'$. Furthermore, this map $\mathrm{Sec}(M_8) \dashrightarrow N_8'$ lifts to $\mathrm{Sec}(M_8) \dashrightarrow N_8$, and this map can be interpreted geometrically as follows (Theorem 3.4, see Figure 2). Suppose we are given a point of $\mathrm{Sec}(M_8)$ on a line connecting two general points of M_8. This corresponds to two ordered octuples of points on $\mathbb{P}^1$, or equivalently an ordered octuple of points on $\mathbb{P}^1 \times \mathbb{P}^1$. Embedding $\mathbb{P}^1 \times \mathbb{P}^1$ by the Segre map yields 8 points in $\mathbb{P}^3$, and hence a point of N_8. The rational map $\mathrm{Sec}(M_8) \dashrightarrow N_8$ must contract 2 dimensions ($\dim \mathrm{Sec}(M_8) = 11$ while $\dim N_8' = 9$); one is the contraction of the secant line, and the other corresponds to the fact that there is a pencil of quadrics passing through 8 points in $\mathbb{P}^3$.

Although it is not clear from the above description, Theorem 3.4 is the hook on which the rest of the argument hangs.

We conjecture that the interrelationships of Figure 1 can be completed as follows.

Conjecture 1.1. *The skew quintic $\mathcal{Q}$ is the trisecant variety (the union of trisecant* lines*) of N_8'. The divisor contracted to M_8 by the dual map $D' : \mathcal{Q} \dashrightarrow \mathcal{C}$ is the quadrisecant variety (union of 4-secant lines) of N_8'.*

As evidence, note that the trisecant variety to N_8' lies in the skew quintic, by Bezout's theorem, even though a naive dimension count suggests that the trisecants should "easily cover" all of $\mathbb{P}^{13}$. Similarly, Bezout's theorem implies that the

quadrisecant variety to N_8' lies in the contracted divisor (analogous to the above argument showing that secant lines to M_8 are contracted by the dual map), even though a naive dimension count suggests that the quadrisecants should "easily cover" all of $\mathbb{P}^{13}$.

1.3. **Other manifestations of this space, and this graded ring.** The extrinsic and intrinsic geometry of $M_n := (\mathbb{P}^1)^n /\!/_{1^n} \mathrm{SL}(2)$ for small n has special meaning often related to the representation theory of $\mathfrak{S}_n$. For example, M_4 relates to the cross ratio, M_5 is the quintic del Pezzo surface, and the Segre cubic M_6 has well known remarkable geometry (see [HMSV2] for further discussion). The space M_8 may be the last of the M_n with such individual personality. For example, over $\mathbb{C}$, the space may be interpreted as a ball quotient in two ways:

(1) Deligne and Mostow [DM] showed that M_8 is isomorphic to the Satake-Baily-Borel compactification of an arithmetic quotient of the 5-dimensional complex ball, using the theory of periods of a family of curves that are fourfold cyclic covers of $\mathbb{P}^1$ branched at the 8 points.
(2) Kondo [Kon] showed that M_8 may also be interpreted in terms of moduli of certain K3 surfaces, and thus M_8 is isomorphic to the Satake-Baily-Borel compactification of a quotient of the complex 5-ball by $\Gamma(1-i)$, an arithmetic subgroup of a unitary group of a hermitian form of signature $(1,5)$ defined over the Gaussian integers. See also [FS2, p. 12] for details and discussion.

Both interpretations are $\mathfrak{S}_8$-equivariant (see [Kon, p. 8] for the second).

Similarly, the graded ring $R_\bullet(M_8)$ has a number of manifestations:

(1) It is isomorphic to the full ring of modular forms of $\Gamma(1-i)$ [FS2, p. 2], via the Borcherds additive lifting.
(2) It is the space of sections of multiples of a certain line bundle on $\overline{\mathcal{M}}_{0,8}$ (as there is a morphism $\overline{\mathcal{M}}_{0,8} \to M_8$, [Ka], see also [AL]).
(3) Igusa [I] showed that there is a natural (non-surjective) map $A(\Gamma_3[2])/\mathcal{I}_3[2]^0 \to R_\bullet(M_8)$, where $A(\Gamma_3[2])$ is the ring of Siegel modular forms of weight 2 and genus 3. (See [FS2, §3] for more discussion.)
(4) It is a quotient of the third in a sequence of algebras related to the orthogonal group $\mathrm{O}(2m, \mathbb{F}_2)$ defined by Freitag and Salvati Manni, see [FS1], [FS2, §2]. (The cases $m = 5$ and $m = 6$ are related to Enriques surfaces.)

One reason for M_8 to be special is the coincidence $\mathfrak{S}_8 \cong \mathrm{O}(6, \mathbb{F}_2)$. A geometric description of this isomorphism in this context is given in [FS2, §4]. Another reason is Deligne and Mostow's table [DM, p. 86].

1.4. **Miscellaneous facts about M_8 and N_8.** We collect miscellaneous facts about M_8 and N_8 in case they prove useful. The graded free resolution is given in Proposition 7.2. The Hilbert function $f(k) = \dim R_k(M_8)$ follows from this, but was computed classically (see for example [Ho, p. 155, §5.4.2.3]):

$$f(k) = \tfrac{1}{3}\left(k^5 + 5k^4 + 11k^3 + 13k^2 + 9k + 3\right)$$

(note this is the same as the Hilbert polynomial), from which the Hilbert series $\sum_{k=0}^{\infty} f(k) t^k$ is

$$\frac{1 + 8t + 22t^2 + 8t^3 + t^4}{(1-t)^6}. \tag{1}$$

(Both formulas are given in [FS2, p. 7].) The degree of M_8 is 40 (the sum of the coefficients of the numerator, or by the method of [HMSV1, p. 190]). Of course M_8 is projectively normal, by what is sometimes called the first fundamental theorem of invariant theory. It is arithmetically Gorenstein, as the numerator of the Hilbert series is symmetric [BH, Corollary 4.4.6]. Thus the a-invariant is -2 (see Proposition 7.1). It doesn't satisfy the N_2 condition of Green and Lazarsfeld: from the minimal graded free resolution of §7 the 14 quadric relations have nonlinear syzygies. It is not Koszul (as the dual Hilbert series $1/H(-t)$ has negative coefficients, and for Koszul algebras this cannot happen, see for example [P, equ. (1)]).

By computer calculation, one may show that the Hilbert series for N_8 is

$$\frac{1+4t+31t^2+40t^3+31t^4+4t^5+t^6}{(1-t)^{10}},$$

from which we see that N_8 is arithmetically Gorenstein, and the a-invariant is -4. Another way to see that N_8 is Gorenstein is to apply a result of F. Knop [Kn] that given a linear action of a group on affine space that preserves volume (i.e. it is a subgroup of SL), such that the unstable locus has codimension at least 2, the subring of invariants is Gorenstein. One may similarly compute that the Hilbert series for N_8' is

$$\frac{1+4t+10t^2+20t^3+21t^4}{(1-t)^{10}}$$

from which $\deg N_8' = 56$.

1.5. **Relation to the six-point case.** (We will not need this picture, so we omit all details.) The classical geometry of six points in projective space, Figure 3, shows strong similarities to Figure 1. This can be made more precise in a number of ways. Here is one way to see Figure 3 "at the boundary" of Figure 1. In the space of 8 points in $\mathbb{P}^3$ (the bottom left of Figure 1), consider the locus where the two given points (of the eight) coincide. Projecting from that point of $\mathbb{P}^3$, the remaining six points (generally) give six points in $\mathbb{P}^2$ (the bottom left of Figure 3). This can be extended to all parts of the two Figures, in a way respecting the Gale and projective dualities.

1.6. **Acknowledgments.** Foremost we thank Igor Dolgachev, who predicted the existence of the cubic of $\mathcal{C}$ to us. Diane Maclagan and Greg Smith gave essential advice on computational issues at key points in this project. We also thank Daniel Erman, Sam Grushevsky, Shrawan Kumar, Riccardo Salvati Manni, and Larry O'Neil for helpful comments.

2. Preliminaries on invariant theory and representation theory

2.1. **Invariants of n points in $\mathbb{P}^{m-1}$ (with linearization $1,\dots,1$).** (See [D] for a thorough introduction to all invariant theory facts we need.) The degree d invariants of n points in $\mathbb{P}^{m-1}$ are generated (as a vector space over a ground field $\mathbf{k}$, or more generally as a module over a ground ring) by invariants corresponding to certain tableaux: $m \times (dn/m)$ matrices, with entries consisting of the numbers 1 through n, each appearing d times. To such a tableau, we associate a product of $m \times m$ determinants, one for each column. To each column, we associate the $m \times m$ determinant whose ith row consists of the projective coordinates of the point

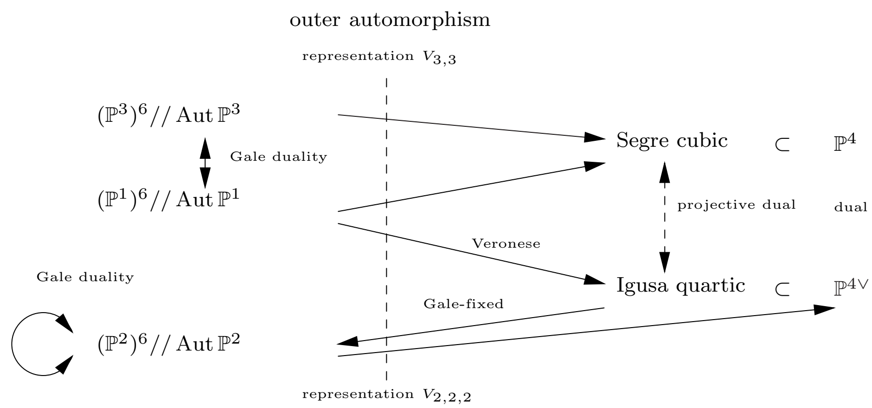

Figure 3. The classical geometry of six points in projective space (cf. Figure 1).

indexed by the entry in that row. For example, if $m = d = 2$ and $n = 4$, and the four points in $\mathbb{P}^1$ have coordinates $[x_i : y_i]$ $(1 \leq i \leq 4)$, then corresponding to

1	2	1	4
3	3	4	2

we associate the SL(2)-invariant

$$(x_1y_3 - x_3y_1)(x_2y_3 - x_3y_2)(x_1y_4 - x_4y_1)(x_4y_2 - x_2y_4).$$

The linear relations among these invariants are spanned by three basic types: (i) columns can be rearranged without changing the invariant (obvious); (ii) swapping two entries in the same column changes the sign of the invariant (obvious); and (iii) Plücker or straightening relations, which we do not describe here (see [HMSV4, §1.3] for a graphical description). The straightening algorithm implies that for fixed n, m, d, the semistable tableaux (where the entries are increasing vertically and weakly increasing horizontally) form a basis.

If $m = 2$ and n is even, it is not hard to see (and a theorem of Kempe, see for example [HMSV1, Thm. 2.3]) that the ring of invariants is generated in degree 1. Thus the GIT quotient $(\mathbb{P}^1)^n//\mathrm{SL}(2)$ naturally comes with a projective embedding, whose coordinates correspond to $2 \times (n/2)$ tableaux. It is helpful to interpret the invariants as directed graphs on n vertices, where for each column $\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}$ we draw an edge $\vec{ij}$ (see [HMSV4, §1.2]). In this language, there is a basis consisting of upwards-oriented non-crossing graphs (those graphs with only edges $\vec{ij}$ with $j > i$, where when represented with the vertices cyclically arranged around a circle, no two edges cross). This basis is different than the one provided by semi-standard tableaux. As an example, Figure 4 gives a basis for $R_1(M_8)$. The following information is omitted to highlight the symmetries: the vertices are labeled cyclically 1 through 8 (it does not matter to us where one starts), and edges are upwards-oriented (if $i < j$, edge ij is oriented $\vec{ij}$).

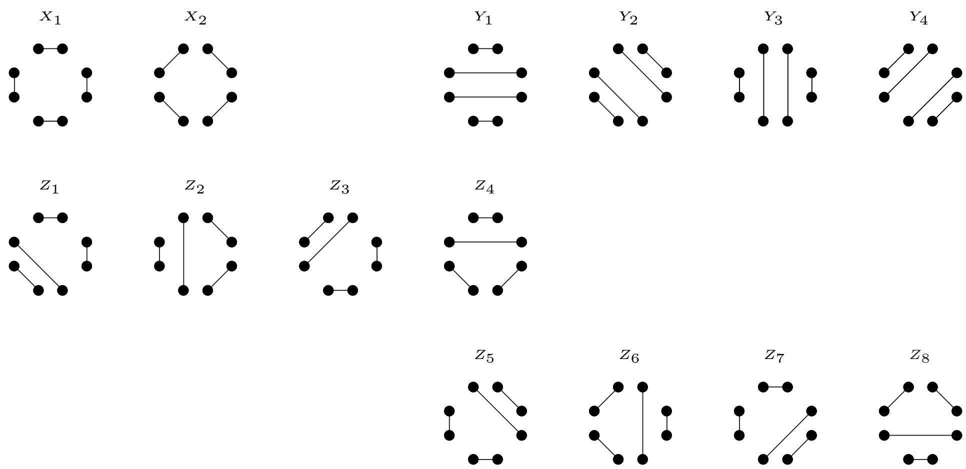

FIGURE 4. The fourteen non-crossing matchings on eight points.

If m is arbitrary, $d = 1$, and n is divisible by m, the description of the degree 1 invariants, with its $\mathfrak{S}_n$ action, is precisely the usual tableaux description of the irreducible $\mathfrak{S}_n$-representation $V_{(n/m)^m}$. If $n = 8$ and $m = 2$ or $m = 4$, the corresponding representation has dimension 14 (see Fig. 4 for the former), so $\dim R_1(M_8) = \dim R_1(N_8) = 14$.

If $n = 8$ and $m = 2$, we have the quadratic relation

$$\begin{array}{|c|c|c|c|}\hline 1&3&5&7\\\hline 2&4&6&8\\\hline\end{array} \times \begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4&7&8\\\hline\end{array} = \begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4&7&8\\\hline\end{array} \times \begin{array}{|c|c|c|c|}\hline 1&2&5&7\\\hline 3&4&6&8\\\hline\end{array}.$$

(By "relation" we mean the difference between the two sides is an element of $\mathrm{Sym}^2(R_1(M_8))$ mapping to 0 in $R_2(M_8)$.) This is clearly a relation: each column appears the same number of times on each side. All four tableaux are semistandard, so this equation is non-zero. This is an example of a *simple binomial relation*, central to [HMSV4]. With appropriate labelling of vertices, in terms of the variables of Figure 4, the relation is

$$X_2 Y_1 = Z_4 Z_8. \tag{2}$$

2.2. **Representation-theoretic preliminaries: $\mathfrak{S}_8$-decomposition of ideals.** Recall that we are working over a field $\mathbf{k}$ of characteristic 0 (although all statements hold over $\mathbb{Z}[1/8!]$). We will repeatedly use Schur–Weyl duality: for a vector space W and a positive integer n, we have a canonical decomposition

$$W^{\otimes n} = \bigoplus_{\lambda} S_\lambda(W) \otimes V_\lambda,$$

where the sum is over partitions λ of n and S_λ denotes the Schur functor associated to λ. This decomposition is compatible with the commuting actions of $\mathfrak{S}_n$ and $\mathrm{GL}(W)$ on each side. If λ has more parts than the dimension of W then $S_\lambda(W) = 0$, so one can restrict the sum to those partitions having at most $\dim W$ parts.

For such partitions, the spaces $S_\lambda(W)$ form mutually non-isomorphic irreducible representations of $\mathrm{GL}(W)$. As an example, let $n = 8$ (which will be the case throughout this paper) and let $W = \mathbb{C}^2$. Let $\lambda = (a, b)$ be a partition of 8 into two parts ($b \leq a$ by convention). The $\mathrm{GL}(W)$-representation $S_\lambda(W)$ is isomorphic to $(\det W)^b \otimes (\mathrm{Sym}^{a-b} W)$. This has an $\mathrm{SL}(W)$ invariant if and only if $a = b$, i.e., if $a = b = 4$. We thus see that $R_1(M_8) = (W^{\otimes 8})^{\mathrm{SL}(W)}$ is isomorphic to the $\mathfrak{S}_8$-representation $V_{4,4}$.

The decomposition of $I_d(M_8)$ into irreducible $\mathfrak{S}_8$-representations may be determined as follows:

$$I_d(M_8) = \ker(\mathrm{Sym}^d(R_1(M_8)) \twoheadrightarrow R_d(M_8)),$$

and $\mathrm{Sym}^d(R_1(M_8))$ may be determined from character theory (using the fact that $R_1(M_8)$ carries the representation $V_{4,4}$), and the representation on

$$R_d(M_8) = \Gamma((\mathbb{P}^1)^8, \mathcal{O}_{(\mathbb{P}^1)^8}(d, \ldots, d))^{\mathrm{SL}(2)}$$

can be determined by Schur-Weyl duality.

Similarly, information about the decomposition of $I_d(N_8')$ into irreducible $\mathfrak{S}_8$-representations can be readily determined by the map

$$I_d(N_8') = \ker(\mathrm{Sym}^d(R_1(N_8)) \to R_d(N_8)).$$

Caution: the map $\mathrm{Sym}^d(R_1(N_8)) \to R_d(N_8)$ is *not* in general a surjection — the analogue of Kempe's theorem does not hold (see Propositions 8.1 and 8.2).

The particular facts we need are the following. The first was proved with 8 replaced by arbitrary even n in [HMSV4, Prop. 6.5], but can be verified for $n = 8$ as described above, or using the methods of Proposition 2.2(a) below.

Proposition 2.1. *We work over a characteristic* 0 *field* $\mathbf{k}$*. In the following table, each representation is multiplicity free. The set of irreducible representations it contains corresponds to the given set of partitions.*

$\mathfrak{S}_8$-representation	Set of partitions of 8
$\mathrm{Sym}^2(R_1(M_8))$	at most four parts, all even
$\bigwedge^2 R_1(M_8)$	exactly four parts, all odd
$R_1(M_8)^{\otimes 2}$	union of previous two sets
$R_2(M_8)$	at most three parts, all even
$I_2(M_8)$	exactly four parts, all even

As described in the introduction, it has been checked by brute force computer calculation (by Maclagan, Koike, and Freitag and Salvati Manni) that the quadratic relations generate the ideal of relations, and a pure thought proof is given here (see Corollary 4.2).

Proposition 2.2. *We work over a characteristic* 0 *field* $\mathbf{k}$*. All statements refer to* $\mathfrak{S}_8$*-representations.*

(a) "The skew cubic relation for M_8." *Up to scalar, there is a single skew-invariant in* $\mathrm{Sym}^3 R_1(M_8)$*, and it is a relation, i.e., it lies in*

$$I_3(M_8) = \ker\left(\mathrm{Sym}^3 R_1(M_8) \twoheadrightarrow R_3(M_8)\right).$$

(b) "The fourteen quadratic relations for M_8." *The degree 2 part of the ideal of M_8 is a single representation of type $V_{2,2,2,2}$:*

$$I_2(M_8) = \ker\left(\operatorname{Sym}^2 R_1(M_8) \twoheadrightarrow R_2(M_8)\right) \cong V_{2,2,2,2}.$$

(c) "The skew quintic relation for N_8'." *There is a non-zero skew-invariant relation in $\operatorname{Sym}^5 R_1(N_8)$ vanishing on N_8, i.e.,*

$$I_5(N_8') = \ker\left(\operatorname{Sym}^5 R_1(N_8') \to R_5(N_8')\right)$$

contains a skew-quintic.

(d) "The fourteen quartic relations for N_8'." *There is a representation of type $V_{4,4}$ in the degree 4 part of the ideal of N_8', i.e., in*

$$I_4(N_8') = \ker\left(\operatorname{Sym}^4 R_1(N_8') \to \operatorname{Sym}^4(N_8')\right).$$

We will verify uniqueness in (c) in Proposition 8.3: there is a one non-zero skew quintic relation up to scalar. We will verify uniqueness in (d) in Corollary 3.14: there is precisely one representation of type $V_{4,4}$ in $I_4(N_8')$.

Proof. To prove (a), first verify that $\operatorname{Sym}^3 R_1(M_8)$ has a single sgn component by character theory. Then note that $R_3(M_8)$ has no sgn component: $(\operatorname{Sym}^3(\mathbf{k}^2))^{\otimes 8}$ has no sgn component because by Schur-Weyl duality it contains no $\mathfrak{S}_8$-representation with more than $4 = \dim(\operatorname{Sym}^3(\mathbf{k}^2))$ rows. (Alternatively, as in [HMSV3, §2], use the fact that the Vandermonde has too high degree. As another alternative, an explicit formula for this skew invariant is given in Remark 3.1.)

Part (b) follows from Proposition 2.1. (Alternatively, use the method of part (a).)

Parts (c) and (d) follow from comparing the appropriate representations in $\operatorname{Sym}^d(R_1(N_8'))$ and $R_d(N_8')$ for $d = 4, 5$, using Schur-Weyl duality for $R_d(N_8)$. (The sign representation sgn appears with multiplicity 4 in $\operatorname{Sym}^5(R_1(N_8'))$, but multiplicity 3 in $R_5(N_8)$. The representation $V_{4,4}$ appears with multiplicity 7 in $\operatorname{Sym}^4(R_1(N_8'))$, but multiplicity 6 in $R_4(N_8)$.) Note that we only get bounds on the multiplicities since $\operatorname{Sym}^\bullet(R_1(N_8'))$ does not surject onto $R_\bullet(N_8)$. □

3. The web of relationships between M_8 and N_8, via the skew cubic $\mathcal{C}$ and the skew quintic $\mathcal{Q}$

3.1. **The skew cubic relation $\mathcal{C}$.** Let $\mathcal{C}$ be the skew-invariant cubic of Proposition 2.2(a) (which is unique up to scalar). We also denote the corresponding hypersurface in $\mathbb{P}(R_1(M_8)^*)$ by $\mathcal{C}$. An element λ of $R_1(M_8)^*$ induces a derivation on the ring $\operatorname{Sym}(R_1(M_8))$, which we denote by $\partial/\partial\lambda$ (we think of it as taking a partial derivative). We have a map

$$R_1(M_8)^* \otimes \operatorname{sgn} \to \operatorname{Sym}^2(R_1(M_8)), \qquad \lambda \mapsto \frac{\partial \mathcal{C}}{\partial \lambda}$$

which is $\mathfrak{S}_8$-equivariant. The image is an irreducible representation of type $V_{4,4} \otimes \operatorname{sgn} = V_{2,2,2,2}$, and is therefore equal to $I_2(M_8)$ by Proposition 2.1; in other words, the above map furnishes a natural isomorphism

$$R_1(M_8)^* \otimes \operatorname{sgn} \to I_2(M_8).$$

"The" 14 quadrics are the image of the basis of $R_1(M_8)^*$ dual to that of $R_1(M_8)$ given by the 14 planar graphs. The above discussion shows that the partial derivatives of $\mathcal{C}$ all vanish on M_8. Furthermore, the simple binomial relations necessarily

span the same irreducible representation — by Proposition 2.1, the quadratic relations form an irreducible $\mathfrak{S}_8$-representation.

Thus the fivefold M_8 is contained in the singular locus of $\mathcal{C}$. (As described in §1.2, M_8 *is* the singular locus of $\mathcal{C}$: we establish this in Corollary 4.2, though it also follows by the computer calculations of Maclagan, Koike, and Freitag and Salvati Manni, or by those of [HMSV1, Prop. 2.10]. We will not need this fact in this section.)

Remark 3.1. One can describe the cubic explicitly, in terms of the variables of Figure 4:

$$\begin{aligned}\mathcal{C} \quad = \quad & X_1X_2(X_1+X_2) + X_1X_2(Z_1+Z_2+Z_3+Z_4+Z_5+Z_6+Z_7+Z_8) \\ & -(X_1Y_2Y_4 + X_2Y_3Y_1) + (X_1Z_2Z_6 + X_2Z_3Z_7 + X_1Z_4Z_8 + X_2Z_5Z_1) \\ & +(Y_1Z_2Z_6 + Y_2Z_3Z_7 + Y_3Z_4Z_8 + Y_4Z_5Z_1) - (Z_1Z_2Z_3 + Z_2Z_3Z_4 \\ & +Z_3Z_4Z_5 + Z_4Z_5Z_6 + Z_5Z_6Z_7 + Z_6Z_7Z_8 + Z_7Z_8Z_1 + Z_8Z_1Z_2).\end{aligned}$$

(One can verify directly that $\mathfrak{S}_8$ acts on the expression above via the sgn representation as follows. Cyclically rotating the labels on the eight vertices of the graphs of Figure 4 clearly changes the sign of $\mathcal{C}$. One readily checks by hand that swapping two chosen adjacent labels changes the sign of $\mathcal{C}$, using the Plücker relations once.) The connection to the simple binomials is quite explicit. For example, $\frac{\partial\mathcal{C}}{\partial Y_3} = -X_2Y_1 + Z_4Z_8$ is the simple binomial relation (2).

Remark 3.2. One can also describe the cubic conceptually: it it the sum of the cubes of the 105 matchings on 8 points, each weighted by a sign in a systematic manner. Equivalently, it is the skew-average of the cube of any matching. These constructions clearly give skew-invariant cubics, but it is non-trivial to show that they are non-zero. Details are given in [HMSV5, Prop. 3.1].

Proposition 3.3. *Suppose the ground field* $\mathbf{k}$ *is* $\mathbb{Q}$. *Let* $\mathcal{H}_{\mathcal{C}}$ *be the Hessian of* $\mathcal{C}$ *(the determinant of the* 14×14 *Hessian matrix, degree* 14*). Then (the scheme corresponding to)* $\mathcal{H}_{\mathcal{C}}$ *does not contain* $\mathcal{C}$ *(so* $\deg(\mathcal{H}_{\mathcal{C}} \cap \mathcal{C}) = 42$*), and the irreducible components of* $\mathcal{H}_{\mathcal{C}} \cap \mathcal{C}$ *have degree* 21 *or* 42.

This proof uses the only computer calculation we need in §3. The calculation makes essential use of the fact that $\mathbf{k}$ is $\mathbb{Q}$.

We will later (Corollary 3.12 and Proposition 3.19) deduce that $\mathcal{H}_{\mathcal{C}}$ meets $\mathcal{C}$ along an irreducible subvariety ($\mathrm{Sec}(M_8)$) of degree 21, with multiplicity 2, and that this holds over *any* field $\mathbf{k}$ of characteristic 0.

Proof. We choose a suitable plane $\mathbb{P}^2 \subset \mathbb{P}^{13}$ over $\mathbb{Q}$, and observe by computer that the intersection of $\mathcal{H}_{\mathcal{C}} \cap \mathbb{P}^2$ with $\mathcal{C} \cap \mathbb{P}^2$ is an irreducible degree 21 (dimension 0) subscheme, appearing with multiplicity 2. (Short Macaulay2 code is given at [HMSV6].) The result follows. □

3.2. **The (projective) dual map from $\mathcal{C}$ contracts** $\mathrm{Sec}(M_8)$**.** The cubic $\mathcal{C}$ is naturally a subscheme of $\mathbb{P}(R_1(M_8)^*)$. The dual map $D : \mathcal{C} \dashrightarrow \mathbb{P}(R_1(M_8))$ (sending a smooth point of $\mathcal{C}$ to its tangent space) is the rational map corresponding to the map on rings $\mathrm{Sym}(R_1(M_8)^*) \to \mathrm{Sym}(R_1(M_8))/(\mathcal{C})$ which maps $\lambda \in R_1(M_8)^*$ to $\frac{\partial\mathcal{C}}{\partial\lambda}$. The two representations $R_1(M_8)$ and $N_1(M_8)^*$ of $\mathfrak{S}_8$ differ by the sign character. We therefore have a *canonical* isomorphism $\mathbb{P}(R_1(M_8)) = \mathbb{P}(R_1(N_8)^*)$. We regard

D as mapping to $\mathbb{P}(R_1(N_8)^*)$. Note that D blows up the singular locus of $\mathcal{C}$, which includes M_8.

The dual to the cubic $\mathcal{C}$ is a hypersurface: $\mathcal{H}_{\mathcal{C}} \cap \mathcal{C} \neq \mathcal{C}$ by Proposition 3.3 (this can also be checked easily by hand), so $\mathcal{C}$ is not contracted by the dual map.

As argued in §1.2, Bezout's theorem implies that $\mathrm{Sec}(M_8) \subset \mathcal{C}$, and every secant line to M_8 is contracted by the dual map, so $\mathrm{Sec}(M_8)$ is contained in the exceptional divisor of the dual map $D : \mathcal{C} \dashrightarrow \mathbb{P}(R_1(N_8)^*)$. Note that the construction of N_8 gives a map $N_8 \to \mathbb{P}(R_1(N_8)^*)$. The image is N_8' (by the definition of N_8').

Theorem 3.4. *Under the duality map D, the space $\mathrm{Sec}(M_8)$ maps dominantly to N_8'.*

Before proving the theorem, we introduce an auxiliary map and establish a few of its properties. The Segre map gives an embedding $(\mathbb{P}^1)^8 \times (\mathbb{P}^1)^8 \to (\mathbb{P}^3)^8$, which descends to a rational map $\sigma : M_8 \times M_8 \dashrightarrow N_8$. (We only get a rational map since a pair of stable points in $(\mathbb{P}^1)^8$ need not map to a stable point of $(\mathbb{P}^3)^8$.) The following two lemmas give the properties of this map that we need.

Lemma 3.5. *The map σ is dominant.*

Proof. We may assume $\mathbf{k}$ is algebraically closed. Let x be a general point of N_8, which we regard as 8 general points $x_1, \ldots, x_8$ in $\mathbb{P}^3$. Through these 8 points passes a one parameter family of quadrics (since the space of quadrics in $\mathbb{P}^3$ is 9 dimensional), a generic member Q of which is smooth. The group $\mathrm{SL}(4)$ acts transitively on the smooth quadrics in $\mathbb{P}^3$ — this is equivalent to the fact that any two non-degenerate quadratic forms on $\mathbf{k}^4$ are equivalent. Thus after moving $x_1, \ldots, x_8$ by an element of $\mathrm{SL}(4)$ (which does not affect x), we can assume that these 8 points lie on the image of the Segre map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$. Thus each x_i gives rise to a point (y_i, y_i') on $\mathbb{P}^1 \times \mathbb{P}^1$, and so we get two points $y = (y_1, \ldots, y_8)$ and $y' = (y_1', \ldots, y_8')$ on $(\mathbb{P}^1)^8$. It is clear that $\sigma(y, y') = x$, which proves the lemma. □

For two graded rings $A_\bullet$ and $B_\bullet$, we write $A_\bullet \boxtimes B_\bullet$ for the projective coordinate ring of $\mathrm{Proj}(A_\bullet) \times_{\mathbf{k}} \mathrm{Proj}(B_\bullet)$ — the graded ring whose degree n piece is $A_n \otimes_{\mathbf{k}} B_n$.

Lemma 3.6. *The map σ is induced from a map $\sigma^* : R_\bullet(N_8) \to R_\bullet(M_8) \boxtimes R_\bullet(M_8)$ of graded rings.*

Proof. The Segre embedding $(\mathbb{P}^1)^8 \times (\mathbb{P}^1)^8 \to (\mathbb{P}^3)^8$ lifts to a map

$$(\mathbf{k}^2)^8 \times (\mathbf{k}^2)^8 \to (\mathbf{k}^4)^8.$$

The ring $R_\bullet(M_8) \boxtimes R_\bullet(M_8)$ consists of functions on $(\mathbf{k}^2)^8 \times (\mathbf{k}^2)^8$ which are $\mathrm{SL}(2) \times \mathrm{SL}(2)$ invariant and $(\mathbf{k}^\times)^8 \times (\mathbf{k}^\times)^8$ semi-invariant. The ring $R_\bullet(N_8)$ consists of functions on $(\mathbf{k}^4)^8$ which are $\mathrm{SL}(4)$ invariant and $(\mathbf{k}^\times)^8$ semi-invariant. It is clear that functions of the one kind pullback to those of the other under the above map. This pullback map on functions is σ^*. □

Note that $R_\bullet(M_8) \boxtimes R_\bullet(M_8)$ is a subring of $R_\bullet(M_8) \otimes R_\bullet(M_8)$. (There is a slight change in grading, e.g., $R_1(M_8) \otimes R_1(M_8)$ is degree 1 in the former and degree 2 in the latter.) In what follows, we regard σ^* as mapping to the latter ring.

We now prove the theorem.

Proof of Theorem 3.4. Fix an isomorphism $I_2(M_8) \cong R_1(N_8)$ of $\mathfrak{S}_8$-modules. Consider the diagram of spaces:

$$\begin{tikzcd} \mathrm{Cone}(M_8) \times \mathrm{Cone}(M_8) \arrow[r] \arrow[d] & \mathrm{Cone}(N_8) \arrow[d] \\ R_1(M_8)^* \arrow[r] & I_2(M_8)^* \cong R_1(N_8)^* \end{tikzcd}$$

Here Cone denotes the affine cone of a projective variety. All maps are morphisms of affine schemes, not just rational maps. We now explain the maps. We have a natural inclusion $\mathrm{Cone}(M_8) \subset R_1(M_8)^*$. The left map adds its two components inside the vector space $R_1(M_8)^*$. The right map is the natural map $\mathrm{Cone}(N_8) \to R_1(N_8)^*$ obtained by interpreting elements of $R_1(N_8)$ as functions on $\mathrm{Cone}(N_8)$. The bottom map is the cone on the duality map D; more precisely, it is given by the partial derivatives of $\mathcal{C}$. The top map is $\mathrm{Spec}(\sigma^*)$ (which is not quite the cone on σ, but close).

Under the left map, $\mathrm{Cone}(M_8) \times \mathrm{Cone}(M_8)$ maps dominantly to the cone on $\mathrm{Sec}(M_8)$. The top map is dominant — this follows easily from the dominance of σ. Under the right map $\mathrm{Cone}(N_8)$ maps surjectively to $\mathrm{Cone}(N_8')$. It follows that to prove the theorem it is enough to show that the above diagram commutes up to multiplication by a non-zero scalar.

Consider the diagram on rings corresponding to the above diagram of spaces:

$$\begin{tikzcd} R_\bullet(M_8) \otimes R_\bullet(M_8) & R_\bullet(N_8) \arrow[l] \\ \mathrm{Sym}^\bullet(R_1(M_8)) \arrow[u] & \mathrm{Sym}^\bullet(I_2(M_8)) \cong \mathrm{Sym}^\bullet(R_1(N_8)) \arrow[l] \arrow[u] \end{tikzcd}$$

All maps respect the action of $\mathfrak{S}_8$ and respect the various gradings (if defined correctly: one must regrade the N_8 spaces by a factor of 2). To show that the diagram commutes, it suffices to show that it commutes when restricted to the degree two elements in the bottom right, since they generate those rings. The degree two pieces of the bottom right rings are irreducible representations of $\mathfrak{S}_8$ of type $V_{2,2,2,2}$. The degree two piece of the top left ring is $R_1(M_8)^{\otimes 2} \oplus R_2(M_8)^{\oplus 2}$, which by Proposition 2.1 contains exactly one copy of $V_{2,2,2,2}$. It follows that one of the two maps from the bottom right to the top left is a scalar multiple of the other. Since each map is non-zero, the scalar is non-zero. $\square$

Remark 3.7. The above proof may seem surprising, since we showed that a diagram was commutative without using very much about the maps involved. For instance, we only used three properties of the map $\mathrm{Cone}(M_8) \times \mathrm{Cone}(M_8) \to R_1(M_8)^*$, namely: (1) that it is $\mathfrak{S}_8$-equivariant; (2) that the induced map on rings preserves the grading; and (3) that $I_2(M_8)$ is not contained in the kernel of the map on rings (this was used to conclude that the "left then up" map in the diagram of rings was non-zero). However, these are very strong conditions to place on a map: there is a 2 parameter family of such maps, and they all induce the same rational map $M_8 \times M_8 \to \mathbb{P}(R_1(M_8)^*)$. So we really did use everything about the map!

Remark 3.8. Theorem 3.4 can also be proved by an easy algebraic computation, as follows. Let p and q be two generic points of M_8. We write $p = (p_1, \ldots, p_8)$ and for convenience work in inhomogeneous coordinates, so that p_i is interpreted a $[1; p_i]$

in $\mathbb{P}^1$ (and similarly for q). Let t be a generic number and consider the point $p+tq$ on $\operatorname{Sec}(M_8)$. Evaluating $p+tq$ on (2) (a partial derivative of the cubic, and so one coordinate of the dual map) gives

$$\begin{aligned}
&\left(\begin{array}{|c|c|c|c|}\hline 1&3&5&7\\\hline 2&4&6&8\\\hline\end{array}(1;p_i)_{1\le i\le 8} + t\begin{array}{|c|c|c|c|}\hline 1&3&5&7\\\hline 2&4&6&8\\\hline\end{array}(1;q_i)_{1\le i\le 8}\right)\\
+&\left(\begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4&7&8\\\hline\end{array}(1;p_i)_{1\le i\le 8} + t\begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4&7&8\\\hline\end{array}(1;q_i)_{1\le i\le 8}\right)\\
-&\left(\begin{array}{|c|c|c|c|}\hline 1&2&5&7\\\hline 3&4&6&8\\\hline\end{array}(1;p_i)_{1\le i\le 8} + t\begin{array}{|c|c|c|c|}\hline 1&2&5&7\\\hline 3&4&6&8\\\hline\end{array}(1;q_i)_{1\le i\le 8}\right)\\
+&\left(\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4&7&8\\\hline\end{array}(1;p_i)_{1\le i\le 8} + t\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4&7&8\\\hline\end{array}(1;q_i)_{1\le i\le 8}\right)
\end{aligned}\tag{3}$$

We wish to show that this agrees with the image of (p,q) in N_8' under the Segre map followed by $N_8 \to N_8'$. A coordinate of the image of (p,q) is given by the expression

$$t\begin{array}{|c|c|}\hline 1&5\\\hline 2&6\\\hline 3&7\\\hline 4&8\\\hline\end{array}(1;p_i;q_i;p_iq_i)_{1\le i\le 8}.\tag{4}$$

A short argument or computation shows that the two expressions (3) and (4) — both polynomials in the p_i, q_i and t — are equal. The equality of the other components of the two maps follows from either similar computations, or an appeal to the $\mathfrak{S}_8$-symmetry.

Corollary 3.9. *We have (a)* $\dim \operatorname{Sec}(M_8) = 11$*; and (b)* $\deg \operatorname{Sec}(M_8) = 21$ *or* 42.

Proof. (a) Of course $\dim \operatorname{Sec}(M_8) \leq 2\dim(M_8) + 1 = 11$. For the opposite inequality, note that $\dim N_8 = 9$, and we see that the preimage (under the dominant rational map $\operatorname{Sec}(M_8) \dashrightarrow N_8$) of a general point of N_8 has dimension at least 2: one corresponding to the one-parameter family of quadrics through 8 general points in $\mathbb{P}^3$, and one corresponding to the secant line joining those two points of M_8 (corresponding to the two octuples of points in $\mathbb{P}^1$). Thus $\dim \operatorname{Sec}(M_8) \geq 11$.

Part (b) then follows from Proposition 3.3. Note that Proposition 3.3 assumes the base field $\mathbf{k}$ is $\mathbb{Q}$, but it suffices to show Corollary 3.9 in this case, as degree is preserved by extension of base field. $\square$

We pause to take stock of where we are. We now know that $\mathcal{H}_{\mathcal{C}} \cap \mathcal{C}$, which has degree 42, contains $\operatorname{Sec}(M_8)$ (which has degree 21 or 42) as a component. We will soon (Proposition 3.19) see that $\mathcal{H}_{\mathcal{C}} \cap \mathcal{C}$ contains $\operatorname{Sec}(M_8)$ with multiplicity 2 (and hence that $\deg \operatorname{Sec}(M_8) = 21$).

3.3. **There is a skew quintic relation $\mathcal{Q}$ in $I_\bullet(N_8')$ defining the dual hypersurface to $\mathcal{C}$.** As previously mentioned, we have a canonical isomorphism $\mathbb{P}(R_1(M_8)) = \mathbb{P}(R_1(N_8)^*)$, and so we can regard the dual hypersurface $\mathcal{Q}$ (the reduced image of the dual map) to $\mathcal{C}$ as a subvariety of $\mathbb{P}(R_1(N_8)^*)$. As $\mathcal{Q}$ is reduced, it is the zero locus of a unique (up to scaling) square-free polynomial, which we also denote by $\mathcal{Q}$. We begin our analysis of the dual hypersurface with the following result:

Proposition 3.10. *The degree of $\mathcal{Q}$ is at least* 5.

Proof. Let d be the degree of $\mathcal{Q}$. The map D contracts $\mathrm{Sec}(M_8)$, and hence sends $\mathrm{Sec}(M_8)$ into the singular locus of $\mathcal{Q}$. Let f be a partial derivative of $\mathcal{Q}$. Then f has degree $d-1$ and vanishes on $\mathrm{Sing}(\mathcal{Q})$ but not all of $\mathcal{Q}$. Thus D^*f vanishes on $\mathrm{Sec}(M_8)$ but not $\mathcal{C}$. Now, D^*f has degree $2(d-1)$. But $\deg(\mathrm{Sec}(M_8)) \geq 21$ (Corollary 3.9), so by Bezout's theorem, $\deg((D^*f)\cap\mathcal{C}) = 2(d-1)\times 3 \geq 21$, from which $d \geq 5$. $\square$

We will use the following consequence in §8.

Corollary 3.11. *The ideal $I_\bullet(N_8')$ contains no relations of degree less than* 4.

Proof. Suppose that $f \in I_d(N_8')$ is a non-zero relation of degree $d < 4$. Then D^*f does not vanish on $\mathcal{C}$ by Proposition 3.10. Now D^*f is a polynomial of degree $2d$ so by Bezout's theorem, $\deg(D^*f\cap\mathcal{C}) = 6d$. But f vanishes on N_8', so D^*f vanishes on $f^{-1}N_8' = \mathrm{Sec}(M_8)$, so

$$6d \geq \deg(\mathrm{Sec}(M_8)) \geq 21$$

(from Corollary 3.9), from which $d \geq 4$, yielding a contradiction. $\square$

Corollary 3.12. *We have* $\deg\mathrm{Sec}(M_8) = 21$.

Proof. By Proposition 2.2 (d), there exists a non-zero relation of degree 4. By the same argument as the proof of Corollary 3.11, $6\times 4 \geq \deg(\mathrm{Sec}(M_8))$. The result then follows from Corollary 3.9. $\square$

Proposition 3.13. *There is an isomorphism*

$$\Phi : I_4(N_8') \to R_1(M_8),$$

unique up to scalar, characterized by the following property: if $\mathcal{R} \in I_4(N_8')$ is a quartic relation on N_8' then $D^*\mathcal{R}\cap\mathcal{C}$ is the union (sum of divisors) of $\mathrm{Sec}(M_8)$ *and the intersection of $\mathcal{C}$ with the hyperplane determined by $\Phi(\mathcal{R})$.*

Proof. First note the hypersurface $\mathcal{C}$ is factorial (by a theorem of Grothendieck, [G-SGA2, Exp. XI, 3.14], implying that complete intersections factorial in codimension 3 are factorial — our special case can also be shown by hand using Nagata's criterion for factoriality [E, Lem. 19.20] applied to the explicit description of the cubic of Remark 3.1). Thus all Weil divisors are Cartier. Also, by the Lefschetz hyperplane theorem for Picard groups, $\mathrm{Pic}(\mathbb{P}(R_1(M_8)^*)) \to \mathrm{Pic}(\mathcal{C})$ is an isomorphism [G-SGA2, Exp. XII, Cor. 3.7] (line bundles on complete intersections in $\mathbb{P}^n$ of dimension at most 3 are all restrictions from the ambient projective space). Now $\mathrm{Sec}(M_8)$ is a divisor of degree 21 on $\mathcal{C}$. Thus $\mathrm{Sec}(M_8)$ is the vanishing scheme of some section $s \in \Gamma(\mathcal{C}, \mathcal{O}(7)|_{\mathcal{C}})$ (unique up to scalar). We remark that $\mathfrak{S}_8$ thus acts on s by a character, and hence either by the identity by sign.

We begin by noting that D^* yields a linear map $I_4(N_8') \to \Gamma(\mathcal{C}, \mathcal{O}(8)|_{\mathcal{C}})$. For any element of $I_4(N_8')$, its pullback by D vanishes on $D^*N_8' = \mathrm{Sec}(M_8)$, and thus is divisible by (the effective Cartier divisor) s. Dividing by s yields a map $I_4(N_8') \to \Gamma(\mathcal{C}, \mathcal{O}(1)|_{\mathcal{C}})$. From the long exact sequence associated to

$$0 \to \mathcal{O}_{\mathbb{P}^{14}}(-2) \to \mathcal{O}_{\mathbb{P}^{14}}(1) \to \mathcal{O}(1)|_{\mathcal{C}} \to 0,$$

and $\Gamma(\mathbb{P}^{14}, \mathcal{O}(1)) = R_1(M_8)$, we may identify $\Gamma(\mathcal{C}, \mathcal{O}(1)|_{\mathcal{C}})$ with $R_1(M_8)$. We have thus obtained a map $I_4(N_8') \to R_1(M_8)$, unique up to scalar. It is not the zero map. $\square$

Corollary 3.14. *The space $I_4(N_8')$ is isomorphic to $V_{4,4}$ as a representation of $\mathfrak{S}_8$.*

Proof. We know $I_4(N_8')$ contains a copy of $V_{4,4}$, while the proposition shows that $I_4(N_8')$ is 14 dimensional. □

We remark that as $I_4(N_8')$ and $R_1(M_8)$ are both the representation $V_{4,4}$, the section s appearing in the proof of Proposition 3.13, cutting out $\mathrm{Sec}(M_8)$, must be $\mathfrak{S}_8$-invariant.

Theorem 3.15. *The polynomial $\mathcal{Q}$ is degree 5, is skew-invariant under $\mathfrak{S}_8$, and its derivatives belong to $I_4(N_8')$. Furthermore, it is the unique such polynomial, up to scalars.*

Remark 3.16. Since the derivatives of $\mathcal{Q}$ belong to $I_\bullet(N_8')$, the Euler formula implies that $\mathcal{Q}$ itself belongs to $I_\bullet(N_8')$. (See the proof below.) We will verify that $\mathcal{Q}$ is in fact the *unique* skew quintic relation in Proposition 8.3. One might hope that the skew quintic is the signed sum of fifth powers of Specht coordinates, in analogy with the situation for the cubic (see Remark 3.2). We can show that this signed sum is non-zero by an analogous method, it is unfortunately *not* a relation for N_8.

Proof. The proof of Proposition 3.13 re-interprets the dual map $D' : \mathcal{Q} \dashrightarrow \mathcal{C}$ as coming from the linear system of the "14 quartics relations of N_8'." We now make this precise.

Choose an isomorphisms of $\mathfrak{S}_8$-modules $\alpha_1 : R_1(M_8) \to R_1(N_8)^* \otimes \mathrm{sgn}$. Let $\alpha_2 : R_1(M_8) \to I_4(N_8')$ be inverse to the isomorphism Φ of Proposition 3.13. Let

$$D_1', D_2' : \mathbb{P}(R_1(N_8)^*) \dashrightarrow \mathbb{P}(R_1(M_8)^*)$$

be the maps corresponding to the ring maps $\mathrm{Sym}(R_1(M_8)) \to \mathrm{Sym}(R_1(N_8))$ given by mapping $x \in R_1(M_8)$ to $\frac{\partial \mathcal{Q}}{\partial \alpha_1(x)}$ and $\alpha_2(x)$, respectively.

Then $D_1'|_{\mathcal{Q}}$ is (essentially by definition) the dual map D', and Proposition 3.13 interprets the 14-dimensional family of quartic relations of N_8', after pulling back by D to $\mathcal{C}$ and subtracting the base locus $\mathrm{Sec}(M_8)$, with the 14-dimensional family of sections of $\mathcal{O}_{\mathbb{P}R_1(M_8)}(1)$, so $D_2'|_{\mathcal{C}}$ is also the dual map D'.

Now the quartics in $I_4(N_8')$ have no common divisorial component on the hypersurface $\mathcal{Q}$. (Otherwise their quotient by this divisor would be a lower-degree polynomial, which when pulled back to $\mathcal{C}$, would vanish on $\mathrm{Sec}(M_8)$ but not on all of $\mathcal{C}$, which is impossible by Bezout's theorem.) Thus, since $D_1' = D_2'$ on $\mathcal{Q}$, there must be a non-zero homogeneous polynomial P such that

$$\frac{\partial \mathcal{Q}}{\partial \alpha_1(x)} = P\alpha_2(x) \tag{5}$$

for all $x \in R_1(M_8)$. (A priori, this equality should be taken modulo $\mathcal{Q}$, but the left side has degree less than $\deg \mathcal{Q}$.) By the following lemma, P must be a scalar, and so by scaling α_1 we can assume $P = 1$.

Since P has degree 0, the formula (5) implies that $\mathcal{Q}$ has degree 5. Let $\{x_i\}$ be a basis for $R_1(N_8)$ and let $\{x_i^*\}$ be the dual basis. The Euler formula

$$5\mathcal{Q} = \sum_{i=1}^{14} x_i \frac{\partial \mathcal{Q}}{\partial x_i^*} = \sum_{i=1}^{14} x_i \alpha_2(\alpha_1^{-1}(x_i^*))$$

shows that $\mathcal{Q}$ belongs to $I_5(N_8')$ (since the image of α_2 is $I_4(N_8')$). For $g \in \mathfrak{S}_8$, we have

$$\frac{\partial \mathcal{Q}}{\partial \alpha_1(gx)} = \alpha_2(gx) = g\alpha_2(x) = \frac{\partial(g\mathcal{Q})}{\partial(g\alpha_1(x))} = \operatorname{sgn}(g)\frac{\partial(g\mathcal{Q})}{\partial \alpha_1(gx)}.$$

The appearance of sgn in the above equation comes from the twist by sgn in the definition of α_1. The above equation shows that all the derivatives of $\operatorname{sgn}(g)\mathcal{Q}$ and $g\mathcal{Q}$ agree, which implies that $g\mathcal{Q} = \operatorname{sgn}(g)\mathcal{Q}$, i.e., $\mathcal{Q}$ is skew-invariant under $\mathfrak{S}_8$.

We now verify that $\mathcal{Q}$ is the unique skew-invariant element of $I_5(N_8')$ (up to scalars) whose derivatives belong to $I_4(N_8')$. Thus let $\mathcal{Q}'$ be an arbitrary such element. Let $\beta : R_1(N_8)^* \otimes \operatorname{sgn} \to I_4(N_8')$ be the map $\lambda \mapsto \frac{\partial \mathcal{Q}}{\partial \lambda}$ and let β' be the analogous map for $\mathcal{Q}'$. Because $\mathcal{Q}$ is skew-invariant, the map β is $\mathfrak{S}_8$-equivariant, and similarly for β'. Since $I_4(N_8')$ is irreducible under $\mathfrak{S}_8$, we have $\beta' = c\beta$ for some scalar c. This implies that $\mathcal{Q}' = c\mathcal{Q}$, as was to be shown. □

Lemma 3.17. *Let Q be a square-free homogeneous polynomial of degree > 1 in several variables over a field $\mathbf{k}$. Then the partial derivatives of Q have no common factor of degree ≥ 1.*

Proof. Assume for the sake of contradiction that F is an irreducible polynomial of degree ≥ 1 dividing all the partial derivatives of Q. Necessarily, F is homogeneous. We have $dQ = 0$ on the irreducible hypersurface $F = 0$, and so Q is constant on $F = 0$. Since Q and F are homogeneous, we actually have $Q = 0$ on $F = 0$ and so $Q = FG$ for some homogeneous polynomial G. Let x be an indeterminate appearing in F. Then $\partial_x Q = (\partial_x G)F + (\partial_x F)G$. Since F divides $\partial_x Q$ by assumption and $\partial_x F$ is non-zero and coprime to F, we find that F divides G. This shows that F^2 divides Q, a contradiction. □

3.4. **The singular locus of $\mathcal{Q}$ is N_8'.** The dual map $D' : \mathcal{Q} \dashrightarrow \mathcal{C}$ blows up precisely $\operatorname{Sing} \mathcal{Q}$, which is cut out by the 14 quartics spanning $I_4(N_8')$. The exceptional divisor on $\mathcal{C}$ is the intersection of the pull-back of these quartics under D, which (from the proof of Proposition 3.13) is $\operatorname{Sec}(M_8)$. But $D(\operatorname{Sec}(M_8)) = N_8'$. Thus we have shown the following.

Theorem 3.18. *We have $\operatorname{Sing}(\mathcal{Q}) = N_8'$.*

Also, the Hessian $\mathcal{H}_{\mathcal{C}}$ vanishes precisely along the exceptional divisor, so we can refine Proposition 3.3 to the following.

Proposition 3.19. *The Hessian $\mathcal{H}_{\mathcal{C}}$ vanishes to order 2 along $\operatorname{Sec}(M_8)$.*

Thus the Hessian is a perfect square modulo $\mathcal{C}$. Specifically, if s is the invariant section of $\mathcal{O}(7)|_{\mathcal{C}}$ appearing in the proof of Proposition 3.13, any $\overline{s}$ is any lift of s to a section of $\mathcal{O}(7)$ (which can be taken to be invariant), then the Hessian is $\overline{s}^2$ modulo $\mathcal{C}$.

We make a small remark about another invariant septic. The pullback of the skew quintic $D^*\mathcal{Q}$ to $\mathbb{P}(R_1(M_8)^*)$ has degree 10, and vanishes on the skew cubic $\mathcal{C}$. The residual divisor to $\mathcal{C}$ in $D^*\mathcal{Q}$ is an invariant septic. This septic contains M_8. Because we will not use this fact, we omit the proof.

4. The partial derivatives of $\mathcal{C}$ have no linear syzygies I

The goal of §4–6 is to establish the following:

Theorem 4.1. *The partial derivatives of $\mathcal{C}$ have no linear syzygies.*

We now elaborate on the statement of the theorem. Pick a basis $x_1, \dots, x_{14}$ of $R_1(M_8)$ (we can then make sense of $\frac{\partial \mathcal{C}}{\partial x_i}$ by defining it to be $\frac{\partial \mathcal{C}}{\partial x_i^*}$, where x_i^* is the dual basis). The theorem is then the statement that if $y_1, \dots, y_{14}$ are elements of $R_1(M_8)$ such that $\sum_{i=1}^{14} y_i \frac{\partial \mathcal{C}}{\partial x_i} = 0$ then $y_i = 0$ for all i. We will give a more canonical reformulation of this statement below. In this section, we will reduce the proof of Theorem 4.1 to a problem that we will solve in §6. We first note an important consequence.

Corollary 4.2. *The ideal $I_\bullet(M_8)$ is generated in degree* 2*.*

Another proof, avoiding the complicated toric degeneration of [HMSV4], will be given in §7, see Remark 7.3.

Proof. By [HMSV4, Thm. 5.1], $I_\bullet(M_8)$ is cut out by quadratics and cubics. One can readily check by hand (counting noncrossing graphs) that $\dim(I_3(M_8)) = 14^2$. By Theorem 4.1, the map of 196-dimensional vector spaces $R_1(M_8) \otimes I_2(M_8) \to I_3(M_8)$ has no kernel and is thus surjective. □

Remark 4.3. As remarked in §1.2, Theorem 4.1 holds away from characteristic 3. In characteristic 3, the Euler formula yields a linear syzygy among the 14 quadratic relations, and the skew cubic $\mathcal{C}$ is the remaining generator of the ideal. See [HMSV5, Thm 1.2 and §9] for a proof. This argument requires a computer, unlike the proof of Theorem 4.1.

We prove Theorem 4.1 by the following strategy.

(a) Let $\Psi : \mathfrak{gl}(R_1(M_8)) \to I_3(M_8)$ be the map defined via a natural action of the Lie algebra $\mathfrak{gl}(R_1(M_8)) \cong \mathfrak{gl}(14)$ on $\mathrm{Sym}^3(R_1(M_8))$ (described below). We first observe that the space of linear syzygies between the partial derivatives of $\mathcal{C}$ is exactly $\mathfrak{g} = \ker \Psi$. We note that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{sl}(R_1(M_8))$ and is stable under the action of $\mathfrak{S}_8$.

(b) Next, using general theory developed in §5 concerning G-stable Lie subalgebras of $\mathfrak{sl}(V)$, where V is a representation of G, and the classification of simple Lie algebras, we show that the only $\mathfrak{S}_8$-stable Lie subalgebras of $\mathfrak{sl}(R_1(M_8))$ are 0, $\mathfrak{so}(14)$ and $\mathfrak{sl}(14)$. Thus $\mathfrak{g}$ must be one of these three Lie algebras.

(c) Finally, we show that $\mathfrak{so}(14)$ does not annihilate any non-zero cubic. As $\mathfrak{g}$ is the annihilator of $\mathcal{C}$ (and $\mathfrak{so}(14) \subset \mathfrak{sl}(14)$), we conclude $\mathfrak{g} = 0$.

We now implement this strategy. Consider the composition

$$\begin{aligned}\widetilde{\Psi} : &\mathrm{End}(R_1(M_8)) \otimes \mathrm{Sym}^3(R_1(M_8)) = R_1(M_8) \otimes R_1(M_8)^* \otimes \mathrm{Sym}^3(R_1(M_8)) \to \\ &R_1(M_8) \otimes \mathrm{Sym}^2(R_1(M_8)) \to \mathrm{Sym}^3(R_1(M_8))\end{aligned}$$

where the first map is the partial derivative map and the second map is the multiplication map. One easily verifies that $\widetilde{\Psi}$ is just the map which expresses the action of the Lie algebra $\mathfrak{gl}(R_1(M_8)) = \mathrm{End}(R_1(M_8))$ on the third symmetric power of its standard representation $R_1(M_8)$. We are trying to show that $\widetilde{\Psi}$ induces an injection

$$\Psi : \mathrm{End}(R_1(M_8)) \otimes \mathbf{k}\mathcal{C} \to I_3(M_8).$$

(We know that Ψ maps $\mathrm{End}(R_1(M_8)) \otimes \mathbf{k}\mathcal{C}$ into $I_3(M_8)$ since we know that the partial derivatives of $\mathcal{C}$ belong to $I_2(M_8)$.) Indeed, the kernel of Ψ is the space of linear syzygies between the partial derivatives of $\mathcal{C}$. Now, the kernel of Ψ is equal

to $\mathfrak{g} \otimes \mathbf{k}\mathcal{C}$, where $\mathfrak{g}$ is the annihilator in $\mathfrak{gl}(R_1(M_8))$ of $\mathcal{C}$. Thus Theorem 4.1 is equivalent to the following:

Proposition 4.4. *We have* $\mathfrak{g} = 0$.

We know two important things about $\mathfrak{g}$: first, $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(R_1(M_8))$, as it is the annihilator of some element in a representation of $\mathfrak{gl}(R_1(M_8))$; and second, $\mathfrak{g}$ is stable under $\mathfrak{S}_8$, as the action map Ψ is $\mathfrak{S}_8$-equivariant and $\mathbf{k}\mathcal{C}$ is stable under $\mathfrak{S}_8$. We will prove Proposition 4.4 by first classifying the $\mathfrak{S}_8$-stable Lie subalgebras of $\mathfrak{gl}(R_1(M_8))$ and then proving that $\mathfrak{g}$ cannot be any of them except zero.

Before continuing, we note the following result:

Proposition 4.5. *The Lie algebra* $\mathfrak{g}$ *is contained in* $\mathfrak{sl}(R_1(M_8))$.

Proof. The trace map $\mathfrak{gl}(R_1(M_8)) \to \mathbf{k}$ is $\mathfrak{S}_8$-equivariant, where $\mathfrak{S}_8$ acts trivially on the target $\mathbf{k}$. Thus if $\mathfrak{g}$ contained an element of non-zero trace it would have to contain a copy of the trivial representation. By Proposition 2.1, $\mathfrak{gl}(R_1(M_8)) \cong R_1(M_8)^{\otimes 2}$ is multiplicity free as an $\mathfrak{S}_8$-representation. Thus the one-dimensional space spanned by the identity matrix is the only copy of the trivial representation in $\mathfrak{gl}(R_1(M_8))$. Therefore, if $\mathfrak{g}$ were not contained in $\mathfrak{sl}(R_1(M_8))$ then it would contain the center of $\mathfrak{gl}(R_1(M_8))$. However, we know that the identity matrix does not annihilate $\mathcal{C}$. Thus $\mathfrak{g}$ must be contained in $\mathfrak{sl}(R_1(M_8))$. □

5. Interlude: G-stable Lie subalgebras of $\mathfrak{sl}(V)$

In this section G will denote an arbitrary finite group and V an irreducible representation of G over an algebraically closed field $\mathbf{k}$ of characteristic zero. We investigate the following general problem:

Problem 5.1. *Determine the* G*-stable Lie subalgebras of* $\mathfrak{sl}(V)$.

We do not obtain a complete answer to this question, but we prove strong enough results to determine the answer in our specific situation. We will use the term *G-subalgebra* to mean a G-stable Lie subalgebra.

5.1. **Some structure theory.** Our first result is the following:

Proposition 5.2. *Let V be an irreducible representation of G. Then every solvable G-subalgebra of $\mathfrak{sl}(V)$ is abelian and consists of semi-simple elements.*

Proof. Let $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{sl}(V)$. By Lie's theorem, $\mathfrak{g}$ preserves a complete flag $0 = V_0 \subset \cdots \subset V_n = V$. The action of $\mathfrak{g}$ on each one-dimensional space V_i/V_{i-1} must factor through $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$; thus $[\mathfrak{g},\mathfrak{g}]$ acts by zero on V_i/V_{i-1} and so carries V_i into V_{i-1}. The space $[\mathfrak{g},\mathfrak{g}]V$ is therefore not all of V. On the other hand, $[\mathfrak{g},\mathfrak{g}]$ is G-stable and therefore so is $[\mathfrak{g},\mathfrak{g}]V$. From the irreducibility of V we conclude $[\mathfrak{g},\mathfrak{g}]V = 0$, from which it follows that $[\mathfrak{g},\mathfrak{g}] = 0$. Thus $\mathfrak{g}$ is abelian.

Now let R be the subalgebra of $\mathrm{End}(V)$ generated (under the usual multiplication) by $\mathfrak{g}$. Let R_s (resp. R_n) denote the set of semi-simple (resp. nilpotent) elements of R. Then R_s is a subring of R, R_n is an ideal of R and $R = R_s \oplus R_n$. As $R_n^m = 0$ for some m, the space R_nV is not all of V. As it is G-stable it must be zero, and so $R_n = 0$. We thus find that $R = R_s$ and so all elements of R, and thus all elements of $\mathfrak{g}$, are semi-simple. □

Let V be a representation of G. We say that V is *imprimitive* if there is a decomposition $V = \bigoplus_{i \in I} V_i$ of V into non-zero subspaces, at least two in number, such that each element of G carries each V_i into some V_j. We say that V is *primitive* if it is not imprimitive. Note that primitive implies irreducible. An irreducible representation is imprimitive if and only if it is induced from a proper subgroup.

Proposition 5.3. *Let V be an irreducible representation of G. Then V is primitive if and only if the only abelian G-subalgebra of $\mathfrak{sl}(V)$ is zero.*

Proof. Let V be an irreducible representation of G and let $\mathfrak{g}$ be a non-zero abelian G-subalgebra of $\mathfrak{sl}(V)$. We will show that V is imprimitive. By Proposition 5.2 all elements of $\mathfrak{g}$ are semi-simple. We thus get a decomposition $V = \bigoplus V_\lambda$ of V into eigenspaces of $\mathfrak{g}$ (each λ is a linear map $\mathfrak{g} \to \mathbf{k}$). As $\mathfrak{g}$ is G-stable, each element of G must carry each V_λ into some $V_{\lambda'}$. Note that if $V = V_\lambda$ for some λ then $\mathfrak{g}$ would consist of scalar matrices, which is impossible as $\mathfrak{g}$ is contained in $\mathfrak{sl}(V)$. Thus there must be at least two non-zero V_λ and so V is imprimitive.

We now establish the other direction. Thus let V be an irreducible imprimitive representation of G. We construct a non-zero abelian G-subalgebra of $\mathfrak{sl}(V)$. Write $V = \bigoplus V_i$ where the elements of G permute the V_i. Let p_i be the endomorphism of V given by projecting onto V_i and then including back into V and let $\mathfrak{g}$ be the subspace of $\mathfrak{gl}(V)$ spanned by the p_i. Then $\mathfrak{g}$ is an abelian subalgebra of $\mathfrak{gl}(V)$ since $p_i p_j = 0$ for $i \neq j$. Furthermore, $\mathfrak{g}$ is G-stable since for each i we have $g p_i g^{-1} = p_j$ for some j. Intersecting $\mathfrak{g}$ with $\mathfrak{sl}(V)$ gives a non-zero abelian G-subalgebra of $\mathfrak{sl}(V)$ (the intersection is non-zero because $\mathfrak{g}$ has dimension at least two and $\mathfrak{sl}(V)$ has codimension one). $\square$

We have the following important consequence of Proposition 5.3:

Corollary 5.4. *Let V be a primitive representation of G. Then every G-subalgebra of $\mathfrak{sl}(V)$ is semi-simple.*

Proof. Let $\mathfrak{g}$ be a G-subalgebra of $\mathfrak{sl}(V)$. The radical of $\mathfrak{g}$ is then a solvable G-subalgebra and therefore vanishes. Thus $\mathfrak{g}$ is semi-simple. $\square$

Proposition 5.3 can also be used to give a criterion for primitivity.

Corollary 5.5. *Let V be an irreducible representation of G such that each non-zero G-submodule of $\mathfrak{sl}(V)$ has dimension at least that of V. Then V is primitive.*

Proof. Let $\mathfrak{g}$ be an abelian G-subalgebra of $\mathfrak{sl}(V)$. We will show that $\mathfrak{g}$ is zero. By Proposition 5.2 $\mathfrak{g}$ consists of semi-simple elements and is therefore contained in some Cartan subalgebra of $\mathfrak{sl}(V)$. This shows that $\dim \mathfrak{g} < \dim V$. Thus, by our hypothesis, $\mathfrak{g} = 0$. $\square$

Let V be a primitive G-module and let $\mathfrak{g}$ be a G-subalgebra. As $\mathfrak{g}$ is semi-simple it decomposes as $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ where each $\mathfrak{g}_i$ is a simple Lie algebra. The $\mathfrak{g}_i$ are called the *simple factors* of $\mathfrak{g}$ and are unique. As the simple factors are unique, G must permute them. We call $\mathfrak{g}$ *prime* if the action of G on its simple factors is transitive. Note that in this case the $\mathfrak{g}_i$'s are isomorphic and so $\mathfrak{g}$ is "isotypic." Clearly, every G-subalgebra of $\mathfrak{sl}(V)$ breaks up into a sum of prime subalgebras and so it suffices to understand these.

5.2. **The action of a G-subalgebra on V.** We now consider how a G-stable subalgebra acts on V:

Proposition 5.6. *Let V be a primitive G-module, let $\mathfrak{g}$ be a G-subalgebra of $\mathfrak{sl}(V)$ and let $\mathfrak{g} = \bigoplus_{i \in I} \mathfrak{g}_i$ be the decomposition of $\mathfrak{g}$ into simple factors.*

(1) *The representation of $\mathfrak{g}$ on V is isotypic, that is, it is of the form $V_0^{\oplus m}$ for some irreducible $\mathfrak{g}$-module V_0.*
(2) *We have a decomposition $V_0 = \bigotimes_{i \in I} W_i$ where each W_i is a faithful irreducible representation of $\mathfrak{g}_i$.*
(3) *We have $V_0 \cong V_0^g$ for each element g of G. (Here V_0^g denotes the $\mathfrak{g}$-module obtained by twisting V_0 by the automorphism g induces on $\mathfrak{g}$.)*
(4) *If $\mathfrak{g}$ is a prime subalgebra then for any i and j one can choose an isomorphism $f : \mathfrak{g}_i \to \mathfrak{g}_j$ so that W_i and $f^* W_j$ become isomorphic as $\mathfrak{g}_i$-modules.*

Proof. (1) Since $\mathfrak{g}$ is semi-simple we get a decomposition $V = \bigoplus V_i^{\oplus m_i}$ of V as a $\mathfrak{g}$-module, where the V_i are pairwise non-isomorphic simple $\mathfrak{g}$-modules. Each element g of G must take each isotypic piece $V_i^{\oplus m_i}$ to some other isotypic piece $V_j^{\oplus m_j}$ since the map $g : V \to V^g$ is $\mathfrak{g}$-equivariant. As V is primitive for G, we conclude that it must be isotypic for $\mathfrak{g}$, and so we may write $V = V_0^{\oplus m}$ for some irreducible $\mathfrak{g}$-module V_0.

(2) As V_0 is irreducible, it necessarily decomposes as a tensor product $V_0 = \bigotimes_{i \in I} W_i$ where each W_i is an irreducible $\mathfrak{g}_i$-module. Since the representation of $\mathfrak{g}$ on $V = V_0^{\oplus m}$ is faithful so too must be the representation of $\mathfrak{g}$ on V_0. From this, we conclude that each W_i must be a faithful representation of $\mathfrak{g}_i$.

(3) For any $g \in G$ the map $g : V \to V^g$ is an isomorphism of $\mathfrak{g}$-modules and so $V_0^{\oplus m}$ is isomorphic to $(V_0^{\oplus m})^g = (V_0^g)^{\oplus m}$, from which it follows that V_0 is isomorphic to V_0^g.

(4) Since G acts transitively on the simple factors, given i and j we can pick $g \in G$ such that $g\mathfrak{g}_i = \mathfrak{g}_j$. The isomorphism of V_0 with V_0^g then gives the isomorphism of W_i and W_j as $\mathfrak{g}_i$-modules. □

This proposition gives a strong numerical constraint on prime subalgebras:

Corollary 5.7. *Let V be a primitive representation of G and let $\mathfrak{g} = \mathfrak{g}_0^n$ be a prime subalgebra of $\mathfrak{sl}(V)$, where $\mathfrak{g}_0$ is a simple Lie algebra. Then $\dim V$ is divisible by d^n where d is the dimension of some faithful representation of $\mathfrak{g}_0$. In particular, $\dim V \ge d_0^n$ where d_0 is the minimal dimension of a faithful representation of $\mathfrak{g}_0$.*

5.3. **Self-dual representations.** Let V be an irreducible self-dual G-module. Thus we have a non-degenerate G-invariant form $\langle \cdot, \cdot \rangle : V \otimes V \to \mathbf{k}$. Such a form is unique up to scaling, and either symmetric or anti-symmetric. We accordingly call V *orthogonal* or *symplectic*.

Let A be an endomorphism of V. We define the *transpose* of A, denoted A^t, by the formula

$$\langle A^t v, u \rangle = \langle v, Au \rangle.$$

It is easily verified that $(AB)^t = B^t A^t$ and $({}^g A)^t = {}^g (A^t)$. We call an endomorphism A *symmetric* if $A = A^t$ and *anti-symmetric* if $A = -A^t$. One easily verifies that the commutator of two anti-symmetric endomorphisms is again anti-symmetric. Thus the set of all anti-symmetric endomorphisms forms a G-subalgebra of $\mathfrak{sl}(V)$ which we denote by $\mathfrak{sl}(V)^-$. In the orthogonal case $\mathfrak{sl}(V)^-$ is isomorphic

to $\mathfrak{so}(V)$ as a Lie algebra and $\bigwedge^2 V$ as a G-module, while in the symplectic case it is isomorphic to $\mathfrak{sp}(V)$ as a Lie algebra and $\mathrm{Sym}^2(V)$ as a G-module. We let $\mathfrak{sl}(V)^+$ denote the space of symmetric endomorphisms.

Proposition 5.8. *Let V be an irreducible self-dual G-module. Assume that:*

- $\mathrm{Sym}^2(V)$ *and* $\bigwedge^2 V$ *have no isomorphic G-submodules; and*
- $\mathfrak{sl}(V)^-$ *has no proper non-zero G-subalgebras.*

Then any proper non-zero G-subalgebra of $\mathfrak{sl}(V)$ other than $\mathfrak{sl}(V)^-$ is commutative. In particular, if V is primitive then the G-subalgebras of $\mathfrak{sl}(V)$ are exactly 0, $\mathfrak{sl}(V)^-$ and $\mathfrak{sl}(V)$.

Proof. Let $\mathfrak{g}$ be a non-zero G-subalgebra of $\mathfrak{sl}(V)$. The intersection of $\mathfrak{g}$ with $\mathfrak{sl}(V)^-$ is a G-subalgebra of $\mathfrak{sl}(V)^-$ and therefore either 0 or all of $\mathfrak{sl}(V)^-$. First assume that the intersection is zero. Since the spaces of symmetric and anti-symmetric elements of $\mathfrak{sl}(V)$ have no isomorphic G-submodules, it follows that $\mathfrak{g}$ is contained in the space of symmetric elements of $\mathfrak{sl}(V)$. But two symmetric elements bracket to an anti-symmetric element. Hence all brackets in $\mathfrak{g}$ vanish and so $\mathfrak{g}$ is commutative. Now assume that $\mathfrak{g}$ contains all of $\mathfrak{sl}(V)^-$. It is then a standard fact that $\mathfrak{sl}(V)^-$ is a maximal subalgebra of $\mathfrak{sl}(V)$ and so $\mathfrak{g}$ is either $\mathfrak{sl}(V)^-$ or $\mathfrak{sl}(V)$. (To see this, note that $\mathfrak{sl}(V)=\mathfrak{sl}(V)^-\oplus\mathfrak{sl}(V)^+$ and so to prove the maximality of $\mathfrak{sl}(V)^-$ it suffices to show that $\mathfrak{sl}(V)^+$ is an irreducible representation of $\mathfrak{sl}(V)^-$. In the orthogonal case this amounts to the fact that, as a representation of $\mathfrak{so}(V)$, the space $\mathrm{Sym}^2(V)/W$ is irreducible, where W is the line spanned by the orthogonal form on V. The symplectic case is similar.) □

6. The partial derivatives of $\mathcal{C}$ have no linear syzygies II

We now complete the proof of Theorem 4.1.

Proposition 6.1. *Assume* $\mathbf{k}$ *is algebraically closed. The $\mathfrak{S}_8$-subalgebras of $\mathfrak{sl}(R_1(M_8))$ are exactly 0, $\mathfrak{so}(R_1(M_8))$ and $\mathfrak{sl}(R_1(M_8))$.*

Proof. We begin by noting that any irreducible representation V of any symmetric group $\mathfrak{S}_n$ is defined over $\mathbb{Q}$ and is therefore orthogonal self-dual. Thus $\mathfrak{so}(R_1(M_8))=\mathfrak{sl}(R_1(M_8))^-$ makes sense as an $\mathfrak{S}_8$-subalgebra.

For our particular $\mathfrak{S}_8$-representation $R_1(M_8)$, Proposition 2.1 shows that $\mathrm{Sym}^2(R_1(M_8))$ has five irreducible submodules of dimensions 1, 14, 14, 20 and 56, while $\bigwedge^2 R_1(M_8)$ has two irreducible submodules of dimensions 35 and 56. Furthermore, none of these seven irreducible representations are isomorphic. As all irreducible submodules of $\mathfrak{sl}(R_1(M_8))$ have dimension at least that of $R_1(M_8)$ (which in this case is 14), we see from Corollary 5.5 that $R_1(M_8)$ is primitive. (Note that the one-dimensional representation occurring in $\mathrm{Sym}^2(R_1(M_8))$ is the center of $\mathfrak{gl}(R_1(M_8))$ and does not occur in $\mathfrak{sl}(R_1(M_8))$.)

As $R_1(M_8)$ is primitive, multiplicity free and self-dual, we can apply Proposition 5.8. This shows that to prove the present proposition we need only show that $\mathfrak{so}(R_1(M_8))$ has no proper non-zero $\mathfrak{S}_8$-subalgebras. Thus assume that $\mathfrak{g}'$ is a proper non-zero $\mathfrak{S}_8$-subalgebra of $\mathfrak{so}(R_1(M_8))$. As $\mathfrak{so}(R_1(M_8))=\bigwedge^2 R_1(M_8)$ has two irreducible submodules we see that $\mathfrak{g}'$ must be one of these two irreducible representations. In particular, this shows that $\mathfrak{g}'$ must be prime and so therefore isotypic. By examining the list of simple Lie algebras (see [FH, §9.4]), we see that

there are four isotypic semi-simple Lie algebras of dimension either 35 or 56:

$$\mathfrak{g}_2^4, \qquad \mathfrak{so}(8)^2, \qquad \mathfrak{sl}(3)^7, \qquad \mathfrak{sl}(6).$$

The minimal dimensions of faithful representations of $\mathfrak{g}_2$, $\mathfrak{so}(8)$ and $\mathfrak{sl}(3)$ are 7, 8 and 3. As 7^4, 8^2 and 3^7 are all bigger than $\dim R_1(M_8) = 14$, Corollary 5.7 rules out the first three Lie algebras above. (One can also rule out $\mathfrak{g}_2^4$ and $\mathfrak{sl}(3)^7$ by noting that the alternating group A_8 does not act non-trivially on them.) We rule out $\mathfrak{sl}(6)$ by using Proposition 5.6 and noting that $\mathfrak{sl}(6)$ has no faithful 14-dimensional isotypic representation — this is proved in Lemma 6.2 below. (One can also rule out $\mathfrak{sl}(6)$ by noting that A_8 does not act non-trivially on it.) This shows that $\mathfrak{g}'$ cannot exist, and proves the proposition. □

Lemma 6.2. *The Lie algebra $\mathfrak{sl}(6)$ has exactly two non-trivial irreducible representations of dimension at most 14: the standard representation and its dual. It has no 14-dimensional faithful isotypic representation.*

Proof. For a dominant weight λ let V_λ denote the irreducible representation with highest weight λ. If λ and λ' are two dominant weights then a general fact valid for any semi-simple Lie algebra states

$$\dim V_{\lambda+\lambda'} \geq \max(\dim V_\lambda, \dim V_{\lambda'}).$$

(To see this, recall the Weyl dimension formula:

$$\dim V_\lambda = \prod_{\alpha^\vee > 0} \frac{\langle \lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle},$$

where ρ is half the sum of the positive roots and the product is taken over the positive co-roots $\alpha^\vee$. Then note that $\langle \lambda, \alpha^\vee \rangle$ is positive for any dominant weight λ and any positive co-root $\alpha^\vee$. Thus $\dim V_{\lambda+\lambda'} \geq \dim V_\lambda$.)

Now, let $\varpi_1, \ldots, \varpi_5$ be the fundamental weights for $\mathfrak{sl}(6)$. The representation V_{ϖ_i} is just $\bigwedge^i V$, where V is the standard representation. For $2 \leq i \leq 4$ the space V_{ϖ_i} has dimension ≥ 15. Furthermore, a simple calculation shows that

$$\dim V_{2\varpi_1} = 21, \qquad \dim V_{\varpi_1+\varpi_5} = 168, \qquad \dim V_{2\varpi_5} = 21.$$

(Note that $V_{2\varpi_1}$ is $\mathrm{Sym}^2(V)$, while $V_{2\varpi_5}$ is its dual. This shows why they are 21-dimensional. To compute the dimension of $V_{\varpi_1+\varpi_5}$ we use the formula for the dimension of the relevant Schur functor, [FH, Ex. 6.4].) Thus only V_{ϖ_1} and V_{ϖ_5} have dimension at most 14, and they each have dimension 6. Since 6 does not divide 14 we find that there are no non-trivial 14-dimensional isotypic representations. □

We now have the following:

Proposition 6.3. *The only element of* $\mathrm{Sym}^3(R_1(M_8))$ *annihilated by* $\mathfrak{so}(R_1(M_8))$ *is zero.*

Proof. As mentioned, $R_1(M_8)$ has a non-degenerate symmetric inner product. Pick an orthonormal basis $\{x_i\}$ of $R_1(M_8)$ and let $\{x_i^*\}$ be the dual basis of $R_1(M_8)^*$. We interpret $\mathrm{Sym}^\bullet(R_1(M_8))$ as the polynomial ring in the x_i. The space $\mathfrak{so}(R_1(M_8))$ is spanned by elements of the form $E_{ij} = x_i \otimes x_j^* - x_j \otimes x_i^*$. Recall that, for an element s of $\mathrm{Sym}(R_1(M_8))$, the element $x_i \otimes x_j^*$ of $\mathrm{End}(R_1(M_8))$ takes s to $x_i \frac{\partial s}{\partial x_j}$. Thus we see that s is annihilated by E_{ij} if and only if

$$x_i \frac{\partial s}{\partial x_j} = x_j \frac{\partial s}{\partial x_i}. \tag{6}$$

Therefore s is annihilated by all of $\mathfrak{so}(R_1(M_8))$ if and only if the above equation holds for all i and j.

Let s be an element of $\mathrm{Sym}^3(R_1(M_8))$. We now consider (6) for a fixed i and j. Write

$$s = g_3(x_j) + g_2(x_j)x_i + g_1(x_j)x_i^2 + g_0(x_j)x_i^3$$

where each g_i is a polynomial in x_j whose coefficients are polynomials in the x_k with $k \neq i, j$. Note that g_0 must be a constant by degree considerations. We have

$$x_i \frac{\partial s}{\partial x_j} = g_3'(x_j)x_i + g_2'(x_j)x_i^2 + g_1'(x_j)x_i^3$$

$$x_j \frac{\partial s}{\partial x_i} = x_j g_2(x_j) + 2x_j g_1(x_j)x_i + 3x_j g_0(x_j)x_i^2.$$

We thus find

$$g_2 = 0, \quad 2x_j g_1 = g_3', \quad 3x_j g_0 = g_2', \quad g_1' = 0.$$

From this we deduce that $g_0 = g_2 = 0$ and that g_1 is determined from g_3. The constraint on g_3 is that it must satisfy

$$g_3'(x_j) = x_j g_3''(x_j). \tag{7}$$

Putting

$$g_3(x_j) = a + bx_j + cx_j^2 + dx_j^3$$

we see that (7) is equivalent to $b = d = 0$. We thus have

$$g_3(x_j) = a + cx_j^2, \qquad \text{and} \qquad g_1(x_j) = c$$

and so

$$s = a + c(x_i^2 + x_j^2)$$

is the general solution to (6).

We thus see that if s satisfies (6) for a particular i and j then x_i and x_j occur in s with only even powers. Thus if s satisfies (6) for all i and j then all variables appear to an even power. This is impossible, unless $s = 0$, since s has degree three. Thus we see that zero is the only solution to (6) which holds for all i and j. $\square$

Remark 6.4. The above computational proof can be made more conceptual. By considering the equation (6) for a fixed i and j we are considering the invariants of $\mathrm{Sym}^3(R_1(M_8))$ under a certain copy of $\mathfrak{so}(2)$ sitting inside of $\mathfrak{so}(R_1(M_8))$. The representation $R_1(M_8)$ restricted to $\mathfrak{so}(2)$ decomposes as $S \oplus T$ where S is the standard representation of $\mathfrak{so}(2)$ and T is a 12-dimensional trivial representation of $\mathfrak{so}(2)$. We then have

$$\mathrm{Sym}^3(R_1(M_8))^{\mathfrak{so}(2)} = \bigoplus_{i=0}^{3} \mathrm{Sym}^i(S)^{\mathfrak{so}(2)} \otimes \mathrm{Sym}^{3-i}(T).$$

Finally, our general solution to (6) amounts to the fact that the ring of invariants $\mathrm{Sym}^\bullet(S)^{\mathfrak{so}(2)}$ is generated by the norm form $x_i^2 + x_j^2$.

We can now prove Proposition 4.4, which will establish Theorem 4.1.

Proof of Proposition 4.4. To prove $\mathfrak{g} = 0$ we may pass to the algebraic closure of $\mathbf{k}$; we thus assume $\mathbf{k}$ is algebraically closed. By Proposition 6.1, the Lie algebra $\mathfrak{g}$ must be 0, $\mathfrak{so}(R_1(M_8))$ or $\mathfrak{sl}(R_1(M_8))$. By Proposition 6.3, $\mathfrak{g}$ cannot be $\mathfrak{so}(R_1(M_8))$ or $\mathfrak{sl}(R_1(M_8))$ since it annihilates $\mathcal{C}$, and $\mathcal{C}$ is non-zero. Thus $\mathfrak{g} = 0$. $\square$

7. The minimal graded free resolution of the graded ring of M_8

We now determine the minimal graded free resolution of the invariant ring $R_\bullet(M_8)$. We first review some commutative algebra.

7.1. **Betti numbers of modules over polynomial rings.** Let $P_\bullet$ be a graded polynomial ring over $\mathbf{k}$ in finitely many indeterminates, each of positive degree. Let M be a finitely generated graded $P_\bullet$-module. (To follow our convention of keeping track of graded objects, we should write $M_\bullet$ rather than M. But because we will be resolving M, we do not.)

One can then find a surjection $F \to M$ with F a finite free module having the following property: if $F' \to M$ is another surjection from a finite free module then there is a surjection $F' \to F$ making the obvious diagram commute. This *free envelope* $F \to M$ of M is unique up to non-unique isomorphism.

Build a resolution of M by using free envelopes:

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

Here F_0 is the free envelope of M and F_{i+1} is the free envelope of $\ker(F_i \to F_{i-1})$. Define integers $b_{i,j}$ by

$$F_i = \bigoplus_{j \in \mathbb{Z}} P_\bullet[-i-j]^{\oplus b_{i,j}}.$$

The b_{ij} are called the *Betti numbers* of M and collectively they form the *Betti diagram* of M. They are independent of the choice of free envelopes, as $b_{i,j}$ is also the dimension of the jth graded piece of $\mathrm{Tor}_i^{P_\bullet}(M, P_\bullet/I_\bullet)$, where $I_\bullet$ is ideal of positive degree elements. The Betti numbers have the following properties:

- (B1) We have $b_{i,j} = 0$ for all but finitely many i and j. This follows since each F_i is finitely generated and $F_i = 0$ for i large by Hilbert's syzygy theorem.
- (B2) We have $b_{i,j} = 0$ for $i < 0$. This follows from the definition.
- (B3) If $b_{i_0,j} = 0$ for $j \leq j_0$ then $b_{i,j} = 0$ for all $i \geq i_0$ and $j \leq j_0$. This follows from the fact that if d is the lowest degree occurring in a module M and $F \to M$ is a free envelope then $F_d \to M_d$ is an isomorphism, and thus the lowest degree occurring in $\ker(F \to M)$ is at least $d+1$.
- (B4) In particular, if M is supported in non-negative degrees then $b_{i,j} = 0$ for $j < 0$.
- (B5) Let $f(k) = \dim M_k$ (resp. $g(k) = \dim P_k$) denote the Hilbert function of M (resp. $P_\bullet$). Then

$$f(k) = \sum_{i,j \in \mathbb{Z}} (-1)^i \cdot b_{i,j} \cdot g(k-i-j).$$

  This follows by taking the Euler characteristic of the kth graded piece of $\cdots \to F_1 \to F_0 \to M$.

In particular, if M is supported in non-negative degrees, then its Betti diagram is contained in a bounded subset of the first quadrant.

7.2. **Betti numbers of graded algebras.** Let $R_\bullet$ be a finitely generated graded k-algebra, generated in degree one. Let $P_\bullet = \mathrm{Sym}^\bullet(R_1)$ be the graded polynomial algebra on the first graded piece, so $R_\bullet$ is a $P_\bullet$-module, and we can speak of its Betti numbers of $R_\bullet$ (as a $P_\bullet$-module).

Assume now that the ring $R_\bullet$ is Gorenstein and a domain. The canonical module ω_R of $R_\bullet$ is then naturally a graded module. Furthermore, there exists an integer

a, called the *a-invariant* of $R_\bullet$, such that ω_R is isomorphic to $R_\bullet[a]$. We now have the following important property of the Betti numbers of $R_\bullet$:

(B6) We have $b_{i,j} = b_{r-i,d+a-j}$ where $d = \dim R_\bullet$,

$$r = \dim P_\bullet - \dim R_\bullet = \operatorname{codim}(\operatorname{Spec}(R_\bullet) \subset \operatorname{Spec}(P_\bullet)),$$

and a is the a-invariant of $R_\bullet$.

No doubt this formula appears in the literature, but we derive it here for completeness. We have $\operatorname{Ext}^i_{P_\bullet}(R_\bullet, \omega_{P_\bullet}) \cong \omega_R$ if $i = r$ and 0 if $i \neq r$. If n is the dimension of $P_\bullet$, then $\omega_{P_\bullet} \cong P_\bullet[-n]$. Since $R_\bullet$ is Gorenstein we have $\omega_R \cong R_\bullet[a]$. Therefore we obtain a minimal free resolution $\cdots \to G_1 \to G_0 \to R_\bullet[a] \to 0$ of $R_\bullet[a]$ by $G_i = \operatorname{Hom}_{P_\bullet}(F_{r-i}, P_\bullet[-n])$. Then $\cdots \to G_1[-a] \to G_0[-a] \to R_\bullet \to 0$ is a minimal free resolution of $R_\bullet$, and by uniqueness of the resolution we therefore have $G_i[-a] \cong F_i$ for each i. Now $G_i[-a] \cong \bigoplus_{j'} P_\bullet[-n - r + i + j' - a]$, and so

$$\bigoplus_{j'} P_\bullet[-n + r - i + j' - a]^{b_{r-i,j'}} \cong \bigoplus_{j} P_\bullet[-i - j]^{b_{i,j}}.$$

Equating components of the same degree gives $-n + r - i + j' - a = -i - j$, or $j' = n - r + a - j$. Hence $b_{i,j} = b_{r-i,n-r+a-j} = b_{r-i,d+a-j}$.

7.3. **The minimal graded free resolution of $R_\bullet(M_8)$.** We begin with the following result:

Proposition 7.1. *The ring $R_\bullet(M_8)$ is Gorenstein with a-invariant -2.*

Proof. We recall a theorem of Hochster–Roberts [BH, Theorem 6.5.1]: if V is a representation of the reductive group G (over a field $\mathbf{k}$ of characteristic zero) then the ring of invariants $(\operatorname{Sym}^\bullet V)^G$ is Cohen–Macaulay. As our ring $R_\bullet(M_8)$ can be realized in this manner, with V being the space of 2×8 matrices and $G = \mathrm{SL}(2) \times T$, where T is the maximal torus in $\mathrm{SL}(8)$, we see that $R_\bullet(M_8)$ is Cohen–Macaulay. We next recall a theorem of Stanley [BH, Corollary 4.4.6]: if $R_\bullet$ is a Cohen–Macaulay ring generated in degree one with Hilbert series $f(t)/(1-t)^d$, where d is the Krull dimension of $R_\bullet$, then $R_\bullet$ is Gorenstein if and only if the polynomial f is symmetric. In this case, the a-invariant of $R_\bullet$ is $\deg f - d$. Going back to our situation, the Hilbert series of our ring is given in (1). The numerator is symmetric of degree four and the denominator has degree six. Thus $R_{\bullet(M_8)}$ is Gorenstein with $a = -2$. $\square$

We can now deduce the Betti diagram of $R_\bullet(M_8)$:

Proposition 7.2. *The Betti diagram of $R_\bullet(M_8)$ is given by:*

$j \backslash i$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	14	0	0	0	0	0	0	0
2	0	0	175	512	700	512	175	0	0
3	0	0	0	0	0	0	0	14	0
4	0	0	0	0	0	0	0	0	1

All $b_{i,j}$ not shown are zero.

Proof. We first note that (B6) gives $b_{8-i,4-j} = b_{i,j}$ as $r = 8$, $d = 6$ and $a = -2$. We thus have the symmetry of the table. Now, by (B2) and (B4), $b_{i,j} = 0$ if either i or j is negative. Thus $b_{i,j} = 0$ if $i > 8$ or $j > 4$ by symmetry. Next, observe

that $P_\bullet \to R_\bullet(M_8)$ is the free envelope of $R_\bullet(M_8)$, where $P_\bullet = \mathrm{Sym}^\bullet(R_1(M_8))$. This gives the $i = 0$ column of the table. We now look at the $i = 1$ column. The 14 generators have no linear relations, so $b_{1,0} = 0$. By (B3), $b_{i,0} = 0$ for $i \geq 1$. We also know that there are 14 quadratic relations, so $b_{1,1} = 14$. We now look at the $i = 2$ column of the table. The 14 quadratic relations have no linear syzygies (Theorem 4.1), so $b_{2,1} = 0$. Using (B3) again, we conclude $b_{i,1} = 0$ for $i \geq 2$. We have thus completed the first two rows of the table. The last two rows are then determined by symmetry. The middle row can now be determined from (B5) by evaluating both sides at $k = 2, \ldots, 10$ and solving the resulting upper triangular system of equations for $b_{i,2}$. (In fact, the computation is simpler than that since $b_{i,2} = b_{8-i,2}$ and we know $b_{0,2} = b_{1,2} = 0$, the latter vanishing coming from Corollary 4.2.) □

Remark 7.3. Proposition 7.2 (in particular, the $i = 1$ column of the table) shows that $I_\bullet(M_8)$ is generated by its degree two piece. Thus we have another proof of Corollary 4.2.

Remark 7.4. The resolution of $R_\bullet(M_8)$ as a $P_\bullet$-module, without any consideration of grading, is given by Freitag and Salvati Manni [FS2, Lemma 1.3, Theorem 1.5]. It was obtained by computer.

8. The degree one and two invariants of N_8: generation of $R_\bullet(N_8)$, and representation theory

In this section, we determine part of the structure of the ring $R_\bullet(N_8)$ of invariants of 8 points in $\mathbb{P}^3$ with the assistance of a computer. In particular, we show that this ring is generated in degree 1 and 2, and we determine the actions of Gale duality and $\mathfrak{S}_8$ on the generators. As a consequence, we show that the Gale-invariant invariants are precisely the subring of $R_\bullet(N_8)$ generated in degree 1, and the skew quintic $\mathcal{Q}$ is the unique skew quintic relation in both $R_\bullet(N_8)$ and $R_\bullet(N_8')$.

Proposition 8.1. *The ring of invariants $R_\bullet(N_8)$ is generated in degrees one and two.*

Proof. A filtration of the ring of invariants such that the associated graded ring is the semigroup of Gel'fand-Tsetlin patterns (or equivalently, semistandard tableaux), as described for example by Alexeev and Brion in [AB, §5.1], can be used to show that the ring is generated in degrees at most 4. (Code using the package 4ti2 is available at [HMSV6], but the method is standard.)

To show that the ring is generated in degrees 1 and 2 is then just linear algebra. We compute the dimensions of the subspaces of the degree 3 and 4 pieces generated by the degree 1 and 2 tableaux. Magma code is available at [HMSV6]. (In more detail: We use a Grobner basis for the ideal of Plücker relations. We define a polynomial ring in 70 variables corresponding to the $\binom{8}{4}$ minors of a 4×8 matrix. In a certain term order, the Plücker relations are a Grobner basis for the Plücker ideal, see [MS, Thm. 14.6, p. 277]. We define the monomials corresponding to the Hilbert basis output from the previous 4ti2 program. The point of using the Grobner basis is that one can quickly implement the straightening relations. The normal form of any polynomial will have all semistandard tableaux as monomials. Now, one simply multiplies degree 1 and degree 2 tableaux and then computes the dimension of degree 3, resp. degree 4, subspaces spanned by them.) □

By Corollary 3.11, there are no degree 2 relations for N_8', so $\mathrm{Sym}^2(R_1(N_8)) \to R_2(N_8)$ is an injection.

Proposition 8.2.

(a) *Gale duality acts via the trivial representation on* $R_1(N_8)$.

(b) *Gale duality acts via the sign representation on* $R_2(N_8)/\mathrm{Sym}^2 R_1(N_8)$.

Thus by Proposition 8.1, $N_8' := \mathrm{Proj}\,\mathrm{Sym}^\bullet(R_1(N_8))$ *is the quotient of* N_8 *by Gale-duality.*

Proof. By [HM, Thm. 1.12, p. 690], Gale-duality acts as follows. The column of a tableaux, with rows abcd, is replaced by efgh, where $\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \dots, \mathrm{h}\} = \{1, \dots, 8\}$, and $\mathrm{a} < \mathrm{b} < \mathrm{c} < \mathrm{d}$ and $\mathrm{e} < \mathrm{f} < \mathrm{g} < \mathrm{h}$, with a sign factor of sgn(abcdefgh) (where "abcdefgh" is interpreted as an element of $\mathfrak{S}_8$, i.e. $1 \mapsto a$, $2 \mapsto b$, etc.).

(a) A degree one tableau (a generator of $R_1(N_8)$) has two columns abcd and efgh, which are swapped by this process. As sgn(abcdefgh) = sgn(efghabcd), the sign contributions cancel. Thus every degree one tableau is Gale-invariant.

(b) By enumerating semistandard tableaux, we see that

$$\dim R_2(N_8)/\mathrm{Sym}^2 R_1(N_8) = 21.$$

One readily sees that there are 42 degree 2 tableaux not fixed by the Gale-involution. They come in 21 pairs, and the differences of elements of each pair form a base for the (-1)-eigenspace of the Gale involution. ☐

8.1. **The skew quintic $\mathcal{Q}$ is the unique skew quintic relation.** We can now readily compute the representation of $\mathfrak{S}_8$ on $R_2(N_8)/\mathrm{Sym}^2(R_1(N_8))$, and show that it is isomorphic to the irreducible representation $V_{3,1,1,1,1,1}$. To do this, we analyze $R_2(N_8)$ using Schur-Weyl duality,

$$\oplus_{\lambda \vdash 8} S_\lambda(\mathrm{Sym}^2(C^4)) \otimes V_\lambda,$$

and examine the case $\lambda = (3,1,1,1,1,1)$. We seek the dimension of the SL_4-invariant part of

$$S_{3,1,1,1,1,1}(S_2(\mathbb{C}^4)) = \oplus_{\mu \vdash 16} n_\mu S_\mu(\mathbb{C}^4),$$

where n_μ is the multiplicity of the Schur functor S_μ. The dimension of the SL_4-invariant part is equal to $n_{4,4,4,4}$, which is the number of copies of $S_{4,4,4,4}$ within the plethysm $S_{3,1,1,1,1,1}(S_2(-))$. This can be checked in any number of algebra packages. For example, a calculation in Maple is given in [HMSV6].

Proposition 8.3. *The skew quintic $\mathcal{Q}$ is the unique skew quintic relation in N_8, and hence in N_8'.*

This is now a straightforward verification. As observed in the proof of Proposition 2.2(d), the sign representation appears with multiplicity 4 in $\mathrm{Sym}^5(R_1(N_8))$, and with multiplicity 3 in $R_5(N_8)$. By Theorem 3.15, $\mathcal{Q} \in \ker(\mathrm{Sym}^5(R_1(N_8)) \to R_5(N_8))$. Let W be an $\mathfrak{S}_8$-equivariant lift of $R_2(N_8)/\mathrm{Sym}^2(R_1(N_8))$ to $R_2(N_8)$, so $R_2(N_8) = \mathrm{Sym}^2(R_1(N_8)) \oplus W$ as $\mathfrak{S}_8$ representations, and $W \cong V_{3,1,1,1,1,1}$. By the generation of $R_\bullet(N_8)$ in degrees up to two (Proposition 8.1), we have a *surjection*

$$\mathrm{Sym}^5(R_1(N_8)) \oplus W \otimes \mathrm{Sym}^3(R_1(N_8)) \oplus \left(\mathrm{Sym}^2 W\right) \otimes R_1(N_8) \to R_5(N_8).$$

The result then follows by checking that the sign representation does not appear in $W \otimes \mathrm{Sym}^3(R_1(N_8))$ or $(\mathrm{Sym}^2 W) \otimes R_1(N_8)$, which may be verified using character theory. (See [HMSV6] for maple code.)

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